Exponents, Radicals, and Polynomials

4

Unit Overview

In this unit you will explore multiplicative patterns and representations of nonlinear data. Exponential growth and decay will be the basis for studying exponential functions. You will investigate the properties of powers and radical expressions. You will also perform operations with radical and rational expressions.

Key Terms

As you study this unit, add these and other terms to your math notebook. Include in your notes your prior knowledge of each word, as well as your experiences in using the word in different mathematical examples. If needed, ask for help in pronouncing new words and add information on pronunciation to your math notebook. It is important that you learn new terms and use them correctly in your class discussions and in your problem solutions.

Math Terms

- radical expression
- principal square root
- negative square root
- rationalize
- tree diagram
- geometric sequence
- common ratio
- recursive formula
- exponential growth
- exponential function
- exponential decay
- compound interest
- exponential regression
- polynomial
- degree of a term
- degree of a polynomial
- standard form of a polynomial
- descending order
- leading coefficient
- monomial
- binomial
- trinomial
- difference of two squares
- square of a binomial
- perfect square trinomial
- rational expression

ESSENTIAL QUESTIONS

? How do multiplicative and exponential patterns model the physical world?

? How are adding and multiplying polynomial expressions different from each other?

EMBEDDED ASSESSMENTS

This unit has four embedded assessments, following Activities 21, 23, 25, and 28. They will give you an opportunity to demonstrate what you have learned.

Embedded Assessment 1:

Exponents, Radicals, and Geometric Sequences p. 323

Embedded Assessment 2:

Exponential Functions p. 353

Embedded Assessment 3:

Polynomial Operations p. 383

Embedded Assessment 4:

Factoring and Simplifying Rational Expressions p. 419

UNIT 4

Getting Ready

Write your answers on notebook paper. Show your work.

1. Find the greatest common factor of 36 and 54.
2. List all the factors of 90.
3. Which of the following is equivalent to $39 \cdot 26 + 39 \cdot 13$?

   A. 13^9 **B.** $13^4 \cdot 14$

   C. $13^2 \cdot 3^2 \cdot 2$ **D.** $13^2 \cdot 3^2$
4. Identify the coefficient, base, and exponent of $4x^5$.
5. Explain two ways to evaluate $15(90 - 3)$.
6. Complete the following table to create a linear relationship.

x	2	4	6	8	10
y	3	5			

7. Graph the function described in the table in Item 6.
8. Use ratios to model the following:
   a. 7.5
   b. Caleb receives 341 of the 436 votes cast for class president.

   Students in Mr. Bulluck's Class

Girls	Boys
12	19

   c. girls to boys
   d. boys to total class members
9. Tell whether each number is rational or irrational.
   a. $\sqrt{25}$ **b.** $\frac{4}{3}$
   c. 2.16 **d.** π
10. Calculate.
   a. $\frac{1}{2} + \frac{3}{8}$ **b.** $\frac{5}{12} - \frac{1}{3}$
   c. $\frac{3}{2} \cdot \frac{2}{5}$ **d.** $\frac{5}{8} \div \frac{3}{4}$

Exponent Rules

Icebergs and Exponents

Lesson 19-1 Basic Exponent Properties

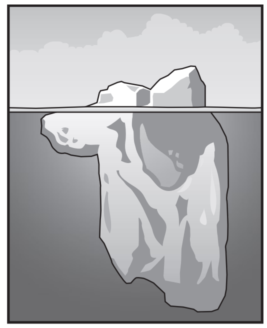
My Notes

Learning Targets:

- Develop basic exponent properties.
- Simplify expressions involving exponents.

SUGGESTED LEARNING STRATEGIES: Create Representations, Predict and Confirm, Look for a Pattern, Think-Pair-Share, Discussion Groups, Sharing and Responding

An *iceberg* is a large piece of freshwater ice that has broken off from a glacier or ice shelf and is floating in open seawater. Icebergs are classified by size. The smallest sized iceberg is called a "growler."



A growler was found floating in the ocean just off the shore of Greenland. Its volume above water was approximately 27 cubic meters.

CONNECT TO GEOLOGY

Because ice is not as dense as seawater, about one-tenth of the volume of an iceberg is visible above water. It is difficult to tell what an iceberg looks like underwater simply by looking at the visible part. Growlers got their name because the sound they make when they are melting sounds like a growling animal.

1. **Reason quantitatively.** Two icebergs float near this growler. One iceberg's volume is 3^4 times greater than the growler. The second iceberg's volume is 2^8 times greater than the growler. Which iceberg has the larger volume? Explain.

2. What is the meaning of 3^4 and 2^8? Why do you think **exponents** are used when writing numbers?

MATH TERMS

The expression 3^4 is a **power**. The **base** is 3 and the **exponent** is 4. The term **power** may also refer to the **exponent**.

3. Suppose the original growler's volume under the water is 9 times the volume above. How much of its ice is below the surface?

4. Write your solution to Item 3 using powers. Complete the equation below. Write the missing terms as a **power** of 3.

$$\text{volume above water} \cdot 3^2 = \text{volume below the surface}$$

$$\square \cdot 3^2 = \square$$

5. Look at the equation you completed for Item 4. What relationship do you notice between the exponents on the left side of the equation and the exponent on the right?

My Notes

6. Use the table below to help verify the pattern you noticed in Item 5. First write each product in the table in expanded form. Then express the product as a single power of the given base. The first one has been done for you.

Original Product	Expanded Form	Single Power
$2^2 \cdot 2^4$	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	2^6
$5^3 \cdot 5^2$		
$x^4 \cdot x^7$		
$a^6 \cdot a^2$		

7. Express regularity in repeated reasoning. Based on the pattern you observed in the table in Item 6, write the missing exponent in the box below to complete the **Product of Powers Property** for exponents.

$$a^m \cdot a^n = a^{\square}$$

8. Use the Product of Powers Property to write $x^{\frac{3}{4}} \cdot x^{\frac{5}{4}}$ as a single power.

CONNECT TO SCIENCE

The formula for density is

$$D = \frac{M}{V}$$

where D is density, M is mass, and V is volume.

9. The density of an iceberg is determined by dividing its mass by its volume. Suppose a growler had a mass of 59,049 kg and a volume of 81 cubic meters. Compute the density of the iceberg.

10. Write your solution to Item 9 using powers of 9.

$$\frac{\boxed{\text{Mass}}}{\boxed{\text{Volume}}} = \boxed{\text{Density}}$$

11. What pattern do you notice in the equation you completed for Item 10?

12. Use the table to help verify the patterns you noticed in Item 11. First write each quotient in the table below in expanded form. Then express the quotient as a single power of the given base. The first one has been done for you.

Original Quotient	Expanded Form	Single Power
$\frac{2^5}{2^2}$	$\frac{2 \bullet 2 \bullet 2 \bullet 2 \bullet 2}{2 \bullet 2} = \frac{\not{2} \bullet \not{2} \bullet 2 \bullet 2 \bullet 2}{\not{2} \bullet \not{2}}$	2^3
$\frac{5^8}{5^6}$		
$\frac{a^3}{a^1}$		
$\frac{x^7}{x^3}$		

13. Based on the pattern you observed in Item 12, write the missing exponent in the box below to complete the **Quotient of Powers Property** for exponents.

$$\frac{a^m}{a^n} = a^{\square}, \text{ where } a \neq 0$$

14. Use the Quotient of Powers Property to write $\frac{a^{\frac{11}{3}}}{a^{\frac{2}{3}}}$ as a single power.

The product and quotient properties of exponents can be used to simplify expressions.

Example A

Simplify: $2x^5 \cdot 5x^4$

Step 1: Group powers with the same base.

$$2x^5 \cdot 5x^4 = 2 \cdot 5 \cdot x^5 \cdot x^4$$

Step 2: Product of Powers Property $= 10x^{5+4}$

Step 3: Simplify the exponent. $= 10x^9$

Solution: $2x^5 \cdot 5x^4 = 10x^9$

My Notes

Example B

Simplify: $\frac{2x^5y^4}{xy^2}$

Step 1: Group powers with the same base. $\frac{2x^5y^4}{xy^2} = 2 \cdot \frac{x^5}{x} \cdot \frac{y^4}{y^2}$

Step 2: Quotient of Powers Property $= 2x^{5-1} \cdot y^{4-2}$

Step 3: Simplify the exponents. $= 2x^4y^2$

Solution: $\frac{2x^5y^4}{xy^2} = 2x^4y^2$

Try These A–B

Simplify each expression.

a. $(4xy^4)(-2x^2y^5)$

b. $\frac{2a^2b^5c}{4ab^2c}$

c. $\frac{6y^3}{18x} \cdot 2xy$

Check Your Understanding

15. Simplify $3yz^2 \cdot 5y^2z$.

16. Simplify $\frac{21f^2g^{\frac{7}{4}}}{7fg^{\frac{3}{4}}}$.

17. A growler has a mass of 243 kg and a volume of 27 cubic meters. Compute the density of the iceberg by completing the following. Write your answer using powers of 3. $\frac{3^5}{3^3} =$

LESSON 19-1 PRACTICE

18. Which expression has the greater value? Explain your reasoning.

a. $2^3 \cdot 2^5$ **b.** $\frac{4^7}{4^3}$

19. The mass of an object is x^{15} grams. Its volume is x^9 cm^3. What is the object's density?

20. The density of an object is y^{10} grams/cm^3. Its volume is y^4 cm^3. What is the object's mass?

21. Simplify the expression $\frac{(3x)^{\frac{1}{3}} \cdot (3x)^{\frac{7}{3}}}{(3x)^{\frac{2}{3}}}$.

22. **Make sense of problems.** Tanika asks Toby to multiply the expression $8^7 \cdot 8^3 \cdot 8^2$. Toby says he doesn't know how to do it, because he believes the Product of Powers Property works with only two exponential terms, and this problem has three terms. Explain how Toby could use the Product of Powers Property with three exponential terms.

MATH TIP

Use a graphic organizer to record the properties of exponents you learn in this activity.

My Notes

Learning Targets:

- Understand what is meant by negative and zero powers.
- Simplify expressions involving exponents.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Discussion Groups, Sharing and Responding, Think-Pair-Share, Close Reading, Note Taking

1. **Attend to precision.** Write each quotient in expanded form and simplify it. Then apply the Quotient of Powers Property. The first one has been done for you.

Original Quotient	Expanded Form	Single Power
$\frac{2^5}{2^8}$	$\frac{2\cdot 2\cdot 2\cdot 2\cdot 2}{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2} = \frac{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{2}}{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot 2\cdot 2\cdot 2} = \frac{1}{2^3}$	$2^{5-8} = 2^{-3}$
$\frac{5^3}{5^6}$		
$\frac{a^3}{a^8}$		
$\frac{x^4}{x^{10}}$		

2. Based on the pattern you observed in Item 1, write the missing exponent in the box below to complete the **Negative Power Property** for exponents.

$$\frac{1}{a^n} = a^{\square}, \text{ where } a \neq 0$$

3. Write each quotient in expanded form and simplify it. Then apply the quotient property of exponents. The first one has been done for you.

Original Quotient	Expanded Form	Single Power
$\frac{2^4}{2^4}$	$\frac{2\cdot 2\cdot 2\cdot 2}{2\cdot 2\cdot 2\cdot 2} = \frac{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{2}}{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{2}} = 1$	$2^{4-4} = 2^0$
$\frac{5^6}{5^6}$		
$\frac{a^3}{a^3}$		

CONNECT TO AP

In calculus, an expression containing a negative exponent is often preferable to one written as a quotient. For example, $\frac{1}{x^3}$ is written x^{-3}.

My Notes

4. Based on the pattern you observed in Item 3, fill in the box below to complete the **Zero Power Property** of exponents.

$$a^0 = \square, \text{ where } a \neq 0$$

5. Use the properties of exponents to evaluate the following expressions.

a. 2^{-3} **b.** $\frac{10^2}{10^{-2}}$ **c.** $3^{-2} \cdot 5^0$ **d.** $(-3.75)^0$

When evaluating and simplifying expressions, you can apply the properties of exponents and then write the answer without negative or zero powers.

Example A

Simplify $5x^{-2}yz^0 \cdot \frac{3x^4}{y^4}$ and write without negative powers.

Step 1: Commutative Property $5x^{-2}yz^0 \cdot \frac{3x^4}{y^4}$

$$= 5 \cdot 3 \cdot x^{-2} \cdot x^4 \cdot y^1 \cdot y^{-4} \cdot z^0$$

Step 2: Apply the exponent rules.

$$= 5 \cdot 3 \cdot x^{-2+4} \cdot y^{1-4} \cdot z^0$$

Step 3: Simplify the exponents.

$$= 15 \cdot x^2 \cdot y^{-3} \cdot 1$$

Step 4: Write without negative exponents.

$$= \frac{15x^2}{y^3}$$

Solution: $5x^{-2}yz^0 \cdot \frac{3x^4}{y^4} = \frac{15x^2}{y^3}$

Try These A

Simplify and write without negative powers.

a. $2a^2b^{-3} \cdot 5ab$ **b.** $\frac{10x^2y^{-4}}{5x^{-3}y^{-1}}$ **c.** $(-3xy^{-5})^0$

My Notes

Check Your Understanding

Simplify each expression. Write your answer without negative exponents.

6. $(z)^{-3}$

7. $12(xyz)^0$

8. $\frac{6^{-4}}{6^{-2}}$

9. $2^3 \cdot 2^{-6}$

10. $\frac{4x^{-2}}{x^3}$

11. $\frac{-5}{(ab)^0}$

LESSON 19-2 PRACTICE

12. For what value of v is $a^v = 1$, if $a \neq 0$?

13. For what value of w is $b^{-w} = \frac{1}{b^9}$, if $b \neq 0$?

14. For what value of y is $\frac{3^3}{3^y} = \frac{1}{9}$?

15. For what value of z is $5^8 \cdot 5^z = 1$?

16. Determine the values of n and m that would make the equation $7^n \cdot 7^m = 1$ a true statement. Assume that $n \neq m$.

17. For what value of x is $\frac{3^x \cdot 2^2}{3^4} = \frac{4}{3}$?

18. Reason abstractly. What is the value of $2^0 \cdot 3^0 \cdot 4^0 \cdot 5^0$? What is the value of any multiplication problem in which all of the factors are raised to a power of 0? Explain.

My Notes

Learning Targets:

- Develop the Power of a Power, Power of a Product, and the Power of a Quotient Properties.
- Simplify expressions involving exponents.

SUGGESTED LEARNING STRATEGIES: Note Taking, Look for a Pattern, Create Representations, Think-Pair-Share, Sharing and Responding, Close Reading

1. Write each expression in expanded form. Then write the expression using a single exponent with the given base. The first one has been done for you.

Original Expression	Expanded Form	Single Power
$(2^2)^4$	$2^2 \cdot 2^2 \cdot 2^2 \cdot 2^2 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	2^8
$(5^5)^3$		
$(x^3)^4$		

2. Based on the pattern you observed in Item 1, write the missing exponent in the box below to complete the **Power of a Power Property** for exponents.

$$(a^m)^n = a^{\square}$$

3. Use the Power of a Power Property to write $\left(x^{\frac{6}{5}}\right)^{25}$ as a single power.

4. Write each expression in expanded form and group like terms. Then write the expression as a product of powers. The first one has been done for you.

Original Expression	Expanded Form	Product of Powers
$(2x)^4$	$2x \cdot 2x \cdot 2x \cdot 2x =$ $2 \cdot 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot x$	2^4x^4
$(-4a)^3$		
$(x^3y^2)^4$		

Lesson 19-3
Additional Properties of Exponents

My Notes

5. Based on the pattern you observed in Item 4, write the missing exponents in the boxes below to complete the **Power of a Product Property** for exponents.

$$(ab)^m = a^{\square} \cdot b^{\square}$$

6. Use the Power of a Product Property to write $\left(c^{\frac{1}{2}}d^{\frac{1}{4}}\right)^8$ as a product of powers.

7. **Make use of structure.** Use the patterns you have seen. Predict and write the missing exponents in the boxes below to complete the **Power of a Quotient Property** for exponents.

$$\left(\frac{a}{b}\right)^m = \frac{a^{\square}}{b^{\square}}, \text{ where } b \neq 0$$

8. Use the Power of a Quotient Property to write $\left(\frac{x^3}{y^6}\right)^{\frac{1}{3}}$ as a quotient of powers.

You can apply these power properties and the exponent rules you have already learned to simplify expressions.

Example A

Simplify $(2x^2y^5)^3\,(3x^2)^{-2}$ and write without negative powers.

Step 1: Power of a Power Property

$$(2x^2y^5)^3\,(3x^2)^{-2} = 2^3x^{2\cdot 3}\,y^{5\cdot 3} \cdot 3^{-2} \cdot x^{2\,\cdot\,-2}$$

Step 2: Simplify the exponents and the numerical terms.

$$= 8 \cdot x^6y^{15} \cdot \frac{1}{3^2} \cdot x^{-4}$$

Step 3: Commutative Property

$$= 8 \cdot \frac{1}{9}x^6 \cdot x^{-4}y^{15}$$

Step 4: Product of Powers Property

$$= \frac{8}{9}x^{6-4}y^{15}$$

Step 5: Simplify the exponents.

$$= \frac{8}{9}x^2y^{15}$$

Solution: $(2x^2y^5)^3(3x^2)^{-2} = \frac{8}{9}x^2y^{15}$

My Notes

Example B

Simplify $\left(\frac{x^2y^{-3}}{z}\right)^2$.

Step 1: Power of a Quotient Property $\left(\frac{x^2y^{-3}}{z}\right)^2 = \frac{x^{2\cdot 2}y^{-3\cdot 2}}{z^2}$

Step 2: Simplify the exponents. $= \frac{x^4y^{-6}}{z^2}$

Step 3: Negative Power Property $= \frac{x^4}{y^6z^2}$

Solution: $\left(\frac{x^2y^{-3}}{z}\right)^2 = \frac{x^4}{y^6z^2}$

Try These A–B

Simplify and write without negative powers.

a. $(2x^2y)^3(-3xy^3)^2$

b. $-2ab(5b^2c)^3$

c. $\left(\frac{4x}{y^3}\right)^{-2}$

d. $\left(\frac{5x}{y}\right)^2\left(\frac{y^3}{10x^2}\right)$

e. $(3xy^{-2})^2(2x^3yz)(6yz^2)^{-1}$

Check Your Understanding

Simplify each expression. Write your answer without negative exponents.

9. $(4x^3y^{-1})^2$

10. $\left(\frac{5x}{y^2}\right)^3$

11. $(-2a^2b^{-2}c)^3(3ab^4c^5)(xyz)^0$

12. $(4fg^3)^{-2}(-4fg^3h)^2(3gh^4)^{-1}$

13. $\left(\frac{2ab}{a^2b^{-2}}\right)^{-3}$

14. $\left[\left(-7nm^2\right)^{-3}\right]^0$

LESSON 19-3 PRACTICE

Simplify.

15. a. $\left(\frac{2}{3}\right)^2$ **b.** $\left(\frac{2}{3}\right)^{-2}$

16. a. $(3x)^3$ **b.** $(3x)^{-3}$

17. a. $(2^5)^4$ **b.** $(2^5)^{-4}$

18. Model with mathematics. The formula for the area of a square is $A = s^2$, where s is the side length. A square garden has a side length of x^4y. What is the area of the garden?

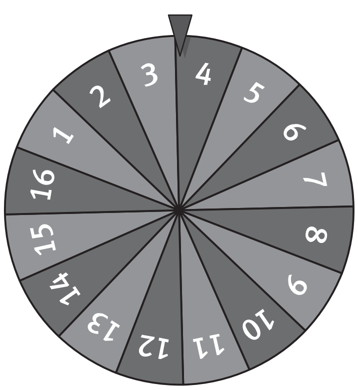
ACTIVITY 19 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 19-1

For Items 1–5, evaluate the expression. Write your answer without negative powers.

1. $x^8 \cdot x^7$
2. $\dfrac{6a^{10}b^9}{3ab^3}$
3. $(6a^2b)(-3ab^3)$
4. $\dfrac{7x^2y^5}{14xy^4}$
5. $\dfrac{2xy^2}{x^5y^3} \cdot \dfrac{5xy^3}{-30y^{-2}}$
6. The volume of an iceberg that is below the water line is 2^5 cubic meters. The volume that is above the water line is 2^2 cubic meters. How many times greater is the volume below the water line than above it?
   A. $2^{2.5}$
   B. 2^3
   C. 2^7
   D. 2^{10}
7. A megabyte is equal to 2^{20} bytes, and a gigabyte is equal to 2^{30} bytes. How many times larger is a gigabyte than a megabyte?
8. A jackpot is worth 10^5 dollars. The contestant who wins the jackpot has the opportunity to put it all on the line with the single spin of a prize wheel. If the contestant spins the number 7 on the wheel, she will win 10^2 times more money. How many dollars will the contestant win if she risks her prize money and spins a 7?



The number of earthquakes of a given magnitude that are likely to occur in any given year is represented by the formula $10^{(8-M)}$, where M is the magnitude. Use this formula for Items 9 and 10.

9. How many earthquakes of magnitude 8 are likely to occur next year?
10. If an earthquake of magnitude 10 occurred last year, how many years will it be before another one of that magnitude is likely to occur?

Lesson 19-2

11. Which of the following expressions is **not** equal to 1?
    A. $x^3 \cdot x^{-3}$ **B.** 1001^0
    C. $\dfrac{a^2b}{ba^2}$ **D.** $\dfrac{y^2}{y^{-2}}$
12. Which of the following expressions is equal to $\dfrac{y}{x^2}$?
    A. $x^{-2}y^3 \cdot y^{-2}$ **B.** $xy^2 \cdot x^{-3}y^{-2}$
    C. $\dfrac{y^2x}{yx^{-3}}$ **D.** $\dfrac{x^2y}{y^{-2}}$

Determine whether each statement is always, sometimes, or never true.

13. For $a \neq 0$, the value of a^{-1} is positive.
14. If n is an integer, then $3^n \cdot 3^{-n}$ equals 1.
15. If $6^p > 0$, then $p > 0$.
16. 4^{-x} equals $\dfrac{1}{4^x}$.
17. If m is an integer, then the value of 2^m is negative.

18. For what value of a is $w^{a-2} = 1$, if $w \neq 0$?

19. For what value of b is $p^{b-1} = \frac{1}{p^5}$, if $p \neq 0$?

For each of the following, give the value of the expression or state that the expression is undefined.

20. x^0 when $x = 0$

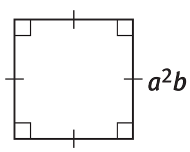
21. 2^{-a} when $a = 0$

22. $\frac{1}{x^p}$ when $x = 0$ and $p > 0$

23. $0^n \cdot 0^{-n}$ when n is an integer

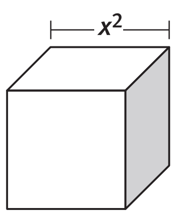
Lesson 19-3

24. The area of a square is given by the formula $A = s^2$, where s is the length of the side. What is the area of the square shown?

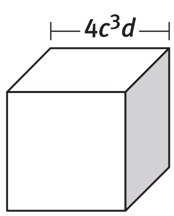


The volume of a cube is given by the formula $V = s^3$, where s is the length of the side. Use this formula for Items 25–27.

25. What is the volume of the cube shown?



26. What is the volume of the cube shown?



27. The volume of a cube is x^{27} cubic inches. What expression represents the length of one side of the cube? Justify your reasoning.

Simplify each expression. Write your answer without negative exponents.

28. $(-5x^2y^{-1})^4$

29. $\left(\frac{c^2d^{-2}}{c}\right)^5$

30. $(x^2y^2z^{-1})^3(xyz^4)(x^3y)$

31. $(m^2n^{-5})^0m^{-7}$

32. $\left(\frac{2x^{-2}}{3}\right)\left(\frac{3x}{4}\right)^2$

33. Which of the following is a true statement about the expression $a^4\left(\frac{1}{a}\right)^2$, given that $a \neq 0$?

A. The expression is always equal to 1.
B. The value of the expression is positive.
C. If a is negative, then the value of the expression is also negative.
D. The expression cannot be simplified any further.

MATHEMATICAL PRACTICES
Construct Viable Arguments and Critique the Reasoning of Others

34. Alana says that $(ab)^3 \cdot (ab)^4$ is the same as $[(ab)^3]^4$. Is Alana correct? Justify your response.

Operations with Radicals

Go Fly a Kite

Lesson 20-1 Radical Expressions

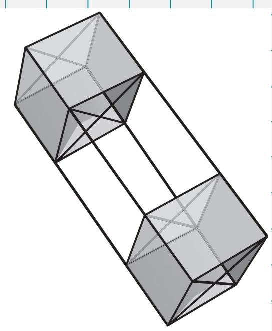
My Notes

Learning Targets:

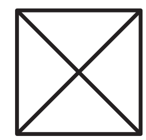
- Write and simplify radical expressions.
- Understand what is meant by a rational exponent.

SUGGESTED LEARNING STRATEGIES: Create Representations, Close Reading, Discussion Groups, Sharing and Responding, Note Taking, Think-Pair-Share

The frame of a box kite has four "legs" of equal length and four pairs of crossbars, all of equal length. The legs of the kite form a square base. The crossbars are attached to the legs so that each crossbar is positioned as a diagonal of the square base.



Top View



1. a. Label the legs of the kite pictured to the right. How many legs are in a kite? How many crossbars?

   b. Label the points on the top view where the ends of the crossbars are attached to the legs A, B, C, and D. Begin at the bottom left and go clockwise.

   c. Use one color to show the sides of the square and another color to show crossbar AC. What two figures are formed by two sides of the square and one diagonal?

Members of the Windy Hill Science Club are building kites to explore aerodynamic forces. Club members will provide paper, plastic, or lightweight cloth for the covering of their kite. The club will provide the balsa wood for the frames.

2. **Model with mathematics.** The science club advisor has created the chart below to help determine how much balsa wood he needs to buy.

   a. For each kite, calculate the exact length of one crossbar that will be needed to stabilize the kite. Use your drawing from Item 1c as a guide for the rectangular base of these box kites.

Kite	Dimensions of Base (in feet)	Exact Length of One Crossbar (in feet)	Kite	Dimensions of Base (in feet)	Exact Length of One Crossbar (in feet)
A	1 by 1		D	1 by 2	
B	2 by 2		E	2 by 4	
C	3 by 3		F	3 by 6	

   b. How much wood would you recommend buying for the crossbars of Kite A? Explain your reasoning.

MATH TIP

Pythagorean Theorem

$$a^2 + b^2 = c^2$$

MATH TIP

If you take the square root of a number that is not a perfect square, the result is a decimal number that does not terminate or repeat and is called an **irrational number**. The exact value of an irrational number must be written using a radical sign.

My Notes

MATH TIP

When there is no root index given, it is assumed to be 2 and is called a *square root*.
$\sqrt{36} = \sqrt[2]{36}$

Each amount of wood in the table in Item 2 is a **radical expression**.

Radical Expression	
An expression of the form $\sqrt[n]{a}$, where a is the radicand, $\sqrt{\ }$ is the radical symbol, and n is the root index.	
$\sqrt[n]{a} = b$, if $b^n = a$.	b is the nth root of a.

Finding the square root of a number or expression is the inverse operation of squaring a number or expression.

$$\sqrt{25} = 5, \text{ because } (5)(5) = 25$$
$$\sqrt{81} = 9, \text{ because } (9)(9) = 81$$
$$\sqrt{x^2} = x, \text{ because } (x)(x) = x^2, x \geq 0$$

Notice also that $(-5)(-5) = (-5)^2 = 25$. The **principal square root** of a number is the positive square root value. The expression $\sqrt{25}$ simplifies to 5, the principal square root. The **negative square root** is the negative root value, so $-\sqrt{25}$ simplifies to -5.

To simplify square roots in which the radicand is not a perfect square:

Step 1: Write the radicand as a product of numbers, one of which is a perfect square.

Step 2: Find the square root of the perfect square.

READING MATH

$a\sqrt{b}$ is read "*a* times the square root of *b*." Example A Part (c) is read "7 times the square root of 12."

Example A

Simplify each expression.

a. $\sqrt{75} = \sqrt{25 \cdot 3} = 5\sqrt{3}$

b. $\sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}$

$\sqrt{72} = \sqrt{9 \cdot 4 \cdot 2} = (3 \cdot 2)\sqrt{2} = 6\sqrt{2}$

c. $7\sqrt{12} = 7\sqrt{4 \cdot 3} = 7(2\sqrt{3}) = 14\sqrt{3}$

d. $\sqrt{c^3} = \sqrt{c^2 \cdot c} = c\sqrt{c}, c \geq 0$

Try These A

Simplify each expression.

a. $\sqrt{18}$ **b.** $5\sqrt{48}$ **c.** $\sqrt{126}$

d. $\sqrt{24y^2}$ **e.** $\sqrt{45b^3}$

My Notes

3. Copy the lengths of the crossbars from the chart in Item 1. Then express the lengths of the crossbars in simplified form.

Kite	Dimensions of Base (feet)	Exact Length of One Crossbar (feet)	Simplified Form of Length of Crossbar
A	1 by 1		
B	2 by 2		
C	3 by 3		
D	1 by 2		
E	2 by 4		
F	3 by 6		

The process of finding roots can be expanded to **cube roots**. Finding the cube root of a number or an expression is the inverse operation of cubing that number or expression.

$$\sqrt[3]{125} = 5, \text{ because } (5)(5)(5) = 125$$
$$\sqrt[3]{y^3} = y, \text{ because } (y)(y)(y) = y^3$$

To simplify cube roots in which the radicand is not a perfect cube, follow the same two-step process that you used for square roots.

Step 1: Write the radicand as a product of numbers, one of which is a perfect cube.

Step 2: Find the cube root of the perfect cube.

MATH TERMS

The root index n can be any integer greater than or equal to 2. A **cube root** has $n = 3$. The cube root of 8 is $\sqrt[3]{8} = 2$ because $2 \cdot 2 \cdot 2 = 8$.

Example B

Simplify each expression.

a. $\sqrt[3]{16} = \sqrt[3]{8 \cdot 2} = \sqrt[3]{8} \cdot \sqrt[3]{2} = 2\sqrt[3]{2}$

b. $3\sqrt[3]{128} = 3\sqrt[3]{64 \cdot 2} = 3\sqrt[3]{64} \cdot \sqrt[3]{2} = 3(4\sqrt[3]{2}) = 12\sqrt[3]{2}$

c. $\sqrt[3]{x^5} = \sqrt[3]{x^3 \cdot x^2} = x\sqrt[3]{x^2}$

Try These B

Simplify each expression.

a. $\sqrt[3]{24}$ **b.** $\sqrt[3]{54z^3}$ **c.** $\sqrt[3]{40b^4}$

Check Your Understanding

4. A kite has a base with dimensions of 2 feet by 3 feet. What is the length of one crossbar that will be needed to stabilize the kite?

5. Simplify.

a. $\sqrt{124}$ **b.** $\sqrt{125d^4}$ **c.** $\sqrt[3]{250}$ **d.** $\sqrt[3]{81m^7}$

Another way to write radical expressions is with fractional exponents.

6. Make use of structure. Use the definition of a radical and the properties of exponents to simplify the expressions of each row of the table. The first row has been done for you.

Radical Form	Simplified Form	Fractional Exponent Form	Simplified Form
$\sqrt{16} \cdot \sqrt{16}$	$4 \cdot 4 = 16$	$16^{\frac{1}{2}} \cdot 16^{\frac{1}{2}}$	$16^{\frac{1}{2}+\frac{1}{2}} = 16^1 = 16$
$\sqrt[3]{8} \cdot \sqrt[3]{8} \cdot \sqrt[3]{8}$		$8^{\frac{1}{3}} \cdot 8^{\frac{1}{3}} \cdot 8^{\frac{1}{3}}$	
$\sqrt[4]{81} \cdot \sqrt[4]{81} \cdot \sqrt[4]{81} \cdot \sqrt[4]{81}$		$81^{\frac{1}{4}} \cdot 81^{\frac{1}{4}} \cdot 81^{\frac{1}{4}} \cdot 81^{\frac{1}{4}}$	
$\sqrt{a} \cdot \sqrt{a}$		$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}}$	

7. Identify and describe any patterns in the table. Write $a^{\frac{1}{n}}$ as a radical expression.

The general rule for fractional exponents when the numerator is not 1 is $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$.

Example C

Write $6^{\frac{2}{3}}$ as a radical expression.

Method 1: $6^{\frac{2}{3}} = 6^{2 \cdot \frac{1}{3}} = (6^2)^{\frac{1}{3}} = \sqrt[3]{6^2}$

Method 2: $6^{\frac{2}{3}} = 6^{\frac{1}{3} \cdot 2} = \left(6^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{6}\right)^2$

Try These C

Write each of the following as a radical expression.

a. $13^{\frac{1}{4}}$ **b.** $7^{\frac{3}{5}}$ **c.** $x^{\frac{3}{2}}$

My Notes

Check Your Understanding

8. a. What is the value of $16^{\frac{1}{4}}$?

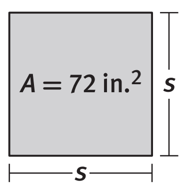
b. What is the value of $16^{\frac{3}{4}}$?

9. For each radical expression, write an equivalent expression with a fractional exponent.

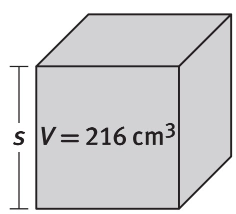
a. $\sqrt{15}$ **b.** $\sqrt[3]{21}$

LESSON 20-1 PRACTICE

10. A square has an area of 72 square inches. What is its side length s? Give the exact answer using simplified radicals.



11. A cube has a volume of 216 cubic centimeters. What is its edge length s? Give the exact answer using simplified radicals.



12. A square has an area of $12x^2$ square feet. What is the length of its sides?

13. A cube has a volume of $128y^3$ cubic millimeters. What is its edge length?

14. A kite has a square base with dimensions of 4 feet by 4 feet. What is the length of one of the diagonal crossbars that will be needed to stabilize the kite?

15. For each radical expression, write an equivalent expression with a fractional exponent.

a. $\sqrt{6}$ **b.** $\sqrt{10}$ **c.** $\sqrt[3]{5}$ **d.** $\sqrt[3]{18}$

16. Reason abstractly. Devise a plan for simplifying the fourth root of a number that is not a perfect fourth power. Explain to a friend how to use your plan to simplify the fourth root. Be sure to include examples.

My Notes

Learning Targets:

- Add radical expressions.
- Subtract radical expressions.

SUGGESTED LEARNING STRATEGIES: Discussion Groups, Close Reading, Note Taking, Think-Pair-Share, Identify a Subtask

The Windy Hill Science Club advisor wants to find the total length of the balsa wood needed to make the frames for the kites. To do so, he will need to add radicals.

MATH TIP

The rational numbers are **closed** under addition and subtraction. This means that the sum or difference of two rational numbers is rational. Similarly, the sum or difference of two irrational numbers is irrational.

Addition Property of Radicals
$a\sqrt{b} \pm c\sqrt{b} = (a \pm c)\sqrt{b}$, where $b \geq 0$.

To add or subtract radicals, the index and radicand must be the same.

Example A

Add or subtract each expression and simplify. State whether the sum or difference is rational or irrational.

a. $3\sqrt{5} + 7\sqrt{5}$
$= (3+7)\sqrt{5}$
$= 10\sqrt{5}$
irrational

←Add or subtract→ the coefficients.

b. $10\sqrt[3]{3} - 4\sqrt[3]{3}$
$= (10-4)\sqrt[3]{3}$
$= 6\sqrt[3]{3}$
irrational

c. $2\sqrt{5} + 8\sqrt{3} + 6\sqrt{5} - 3\sqrt{3}$

Step 1: Group terms with like radicands. $2\sqrt{5} + 6\sqrt{5} + 8\sqrt{3} - 3\sqrt{3}$

Step 2: Add or subtract the coefficients. $= (2+6)\sqrt{5} + (8-3)\sqrt{3}$
$= 8\sqrt{5} + 5\sqrt{3}$

Solution: $2\sqrt{5} + 8\sqrt{3} + 6\sqrt{5} - 3\sqrt{3} = 8\sqrt{5} + 5\sqrt{3}$; irrational

Try These A

Add or subtract each expression and simplify. State whether the sum or difference is rational or irrational.

a. $2\sqrt{7} + 3\sqrt{7} + \frac{2}{3}$

b. $5\sqrt{6} + 2\sqrt{5} - \sqrt{6} + 7\sqrt{5}$

c. $2 + 2\sqrt{2} + \sqrt{8} + 3\sqrt{2}$

My Notes

1. The club advisor also needs to know how much wood to buy for the legs of the kites. Each kite will be 3 feet tall.
   a. Complete the table below.

Kite	Dimensions of Base (feet)	Length of One Crossbar (feet)	Length of One Leg (feet)	Wood Needed for Legs (feet)	Wood Needed for Crossbars (feet)
A	1 by 1				
B	2 by 2				
C	3 by 3				
D	1 by 2				
E	2 by 4				
F	3 by 6				

   b. **Reason quantitatively.** How much balsa wood should the club advisor buy if the club is going to build the six kites described above? Is the result rational or irrational?

   c. Explain how you reached your conclusion.

2. **Use appropriate tools strategically.** Approximately how much balsa wood, in decimal notation, will the club advisor need to buy?
   a. Use your calculator to approximate the amount of balsa wood, and then decide on a reasonable way to round.

   b. Explain why the club advisor would need this approximation rather than the exact answer expressed as a radical.

My Notes

Check Your Understanding

Perform the indicated operations. Be sure to completely simplify your answer. State whether each sum or difference is rational or irrational.

3. a. $7\sqrt{6} + 9\sqrt{6}$ **b.** $12\sqrt{4} - 5\sqrt{4} + 2\sqrt{4}$

4. a. $8\sqrt{3} - \sqrt{12} + 3\sqrt{3}$ **b.** $\sqrt{18} + \sqrt{8} + \sqrt{32}$

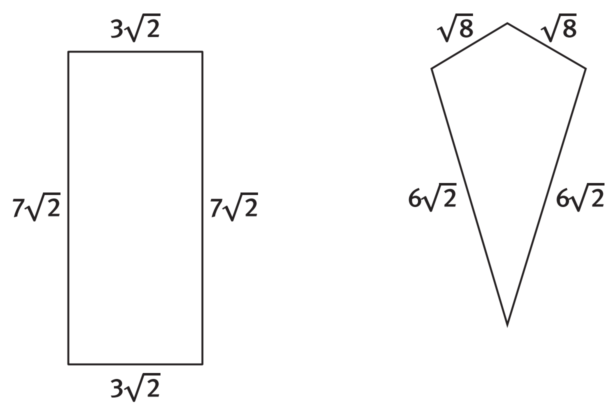
5. a. $\sqrt[3]{16} - \sqrt[3]{2}$ **b.** $\sqrt[3]{3} + \sqrt[3]{24} + 16$

6. a. $3\sqrt{9} + 5\sqrt{9}$ **b.** $3\sqrt{10} + 5\sqrt{10}$

7. Is the sum of a rational number and an irrational number rational or irrational? Support your response with an example.

LESSON 20-2 PRACTICE

Use the figures of a rectangle and kite for Items 8–12.



8. Determine the perimeter of the rectangle.

9. Determine the perimeter of the kite.

10. How much longer is the long side of the rectangle than the longer side of the kite?

11. How much greater is the perimeter of the rectangle than the perimeter of the kite?

12. **Make sense of problems.** How much wood would be required to insert diagonal crossbars in the rectangle?

My Notes

Learning Targets:

- Multiply and divide radical expressions.
- Rationalize the denominator of a radical expression.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Predict and Confirm, Discussion Groups, Close Reading, Marking the Text, Note Taking

1. a. Complete the table below and simplify the radical expressions in the third and fifth columns.

a	b	$\sqrt{a} \cdot \sqrt{b}$	ab	$\sqrt{ab}$
4	9			
100	25			
9	16			

b. Express regularity in repeated reasoning. Use the patterns you observe in the table above to write an equation that relates $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{ab}$.

c. All the values of a and b in part a are perfect squares. In the table below, choose some values for a and b that are *not* perfect squares and use a calculator to show that the equation you wrote in Part (b) is true for those numbers as well.

TECHNOLOGY TIP

Approximate values of square roots that are not perfect squares can be found using a calculator.

a	b	$\sqrt{a} \cdot \sqrt{b}$	ab	$\sqrt{ab}$

d. Simplify the products in Columns A and B below.

A	Simplified Form	B	Simplified Form
$(2\sqrt{4})(\sqrt{9})$		$2\sqrt{4 \cdot 9}$	
$(3\sqrt{4})(5\sqrt{16})$		$(3 \cdot 5)\sqrt{4 \cdot 16}$	
$(2\sqrt{7})(3\sqrt{14})$		$(2 \cdot 3)\sqrt{7 \cdot 14}$	

e. Which products in the table in Part (d) are rational and which are irrational?

My Notes

MATH TIP

The rational numbers are **closed** under multiplication. This means that the product of two rational numbers is rational. Since the coefficients *a* and *c* are rational, their product will also be rational.

MATH TIP

coefficient
index
$5\sqrt[3]{2}$
radicand

CONNECT TO AP

Later in this course, you will study another system of numbers, called the complex numbers. In the complex number system, $\sqrt{-1}$ is defined as the imaginary number *i*.

f. Write a verbal rule that explains how to multiply radical expressions.

Multiplication Property of Radicals
$\left(a\sqrt{b}\right)\left(c\sqrt{d}\right) = ac\sqrt{bd}$, where $b \geq 0$, $d \geq 0$.

To multiply radical expressions, the index must be the same. Find the product of the coefficients and the product of the radicands. Simplify the radical expression.

Example A

Multiply each expression and simplify.

a. $\left(3\sqrt{6}\right)\left(4\sqrt{5}\right) = (3 \cdot 4)\left(\sqrt{6 \cdot 5}\right) = 12\sqrt{30}$

b. $\left(2\sqrt{10}\right)\left(3\sqrt{6}\right)$

$= (2 \cdot 3)\sqrt{10 \cdot 6}$ **Step 1:** Multiply.

$= 6\sqrt{60}$

$= 6\left(\sqrt{4 \cdot 15}\right)$ **Step 2:** Simplify.

$= (6 \cdot 2)\sqrt{15}$

$= 12\sqrt{15}$

c. $\left(2x\sqrt{6x}\right)\left(5\sqrt{3x^2}\right)$

$= 10x\sqrt{6x \cdot 3x^2}$

$= 10x\left(\sqrt{18x^3}\right)$

$= 10x\left(\sqrt{9x^2 \cdot 2x}\right)$

$= (10x)(3x)\left(\sqrt{2x}\right)$

$= 30x^2\sqrt{2x}$

Try These A

Multiply each expression and simplify.

a. $\left(2\sqrt{10}\right)\left(5\sqrt{3}\right)$

b. $\left(3\sqrt{8}\right)\left(2\sqrt{6}\right)$

c. $\left(4\sqrt{12}\right)\left(5\sqrt{18}\right)$

d. $\left(3\sqrt{5a}\right)\left(2a\sqrt{15a^2}\right)$

Division Property of Radicals
$\dfrac{a\sqrt{b}}{c\sqrt{d}} = \dfrac{a}{c}\sqrt{\dfrac{b}{d}}$ where $b \geq 0$, $d \geq 0$.

To divide radical expressions, the index must be the same. Find the quotient of the coefficients and the quotient of the radicands. Simplify the expression.

My Notes

Example B

Divide each expression and simplify.

a. $\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{\frac{6}{2}} = \sqrt{3}$

b. $\frac{2\sqrt{10}}{3\sqrt{2}} = \frac{2}{3}\sqrt{\frac{10}{2}} = \frac{2}{3}\sqrt{5}$

c. $\frac{8\sqrt{24x^2}}{2\sqrt{3}} = \frac{8}{2}\sqrt{\frac{24x^2}{3}} = 4\sqrt{8x^2} = 4\sqrt{4 \cdot 2 \cdot x^2}$

$= 4\left(2x\sqrt{2}\right) = 8x\sqrt{2}$

Try These B

Divide each expression and simplify.

a. $\frac{4\sqrt{42}}{5\sqrt{6}}$

b. $\frac{10\sqrt{54}}{2\sqrt{2}}$

c. $\frac{12\sqrt{75}}{3\sqrt{3}}$

d. $\frac{16\sqrt[3]{8x^{11}}}{8\sqrt[3]{x^2}}$

A radical expression in simplified form does not have a radical in the denominator. Most frequently, the denominator is **rationalized**. You **rationalize the denominator** by simplifying the expression to get a perfect square under the radicand in the denominator.

$$\frac{\sqrt{a}}{\sqrt{b}} \cdot 1 = \left(\frac{\sqrt{a}}{\sqrt{b}}\right)\left(\frac{\sqrt{b}}{\sqrt{b}}\right) = \frac{\sqrt{ab}}{\sqrt{b^2}} = \frac{\sqrt{ab}}{b}$$

MATH TERMS

Rationalize means to make rational. You can **rationalize the denominator** without changing the value of the expression by multiplying the fraction by an appropriate form of 1.

Example C

Rationalize the denominator of $\frac{\sqrt{5}}{\sqrt{3}}$.

Step 1: Multiply the numerator and denominator by $\sqrt{3}$.

$\frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{5}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{15}}{\sqrt{9}}$

Step 2: Simplify.

$= \frac{\sqrt{15}}{3}$

Solution: $\frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{15}}{3}$

CONNECT TO AP

In calculus, both numerators and denominators are rationalized. The procedure for rationalizing a numerator is similar to that for rationalizing a denominator.

Try These C

Rationalize the denominator in each expression.

a. $\frac{\sqrt{11}}{\sqrt{6}}$

b. $\frac{2\sqrt{7}}{\sqrt{5}}$

c. $\frac{3\sqrt{5}}{\sqrt{8}}$

My Notes

Check Your Understanding

Express each expression in simplest radical form. State whether each result in Items 2–5 is rational or irrational.

2. $\left(4\sqrt{7}\right)\left(2\sqrt{3}\right)$

3. $\sqrt{2}\left(\sqrt{2}+3\sqrt{6}\right)$

4. $\frac{\sqrt{75}}{\sqrt{5}}$

5. $\sqrt{\frac{5}{8}}$

6. $\left(3\sqrt{32y}\right)\left(4\sqrt{25y}\right)$

7. $\frac{6\sqrt{98x^4}}{\sqrt{2x}}$

8. Is the product of a nonzero rational number and an irrational number rational or irrational? Support your response with an example.

LESSON 20-3 PRACTICE

Express each expression in simplest radical form. State whether each result in Items 9–12 is rational or irrational.

9. $\left(\sqrt{\frac{1}{2}}\right)\left(\sqrt{\frac{3}{5}}\right)$

10. $\sqrt{27}\bullet\sqrt{\frac{1}{27}}$

11. $\frac{2\sqrt{7}}{\sqrt{3}}$

12. $\frac{4\sqrt{5}}{3\sqrt{2}}$

13. $\left(4\sqrt[3]{4m^2}\right)\left(5m\sqrt[3]{2m^2}\right)$

14. $\frac{2\sqrt{52x^9}}{\sqrt{13x}}$

15. **Attend to precision.** What conditions must be satisfied for a radical expression to be in simplified form?

ACTIVITY 20 PRACTICE

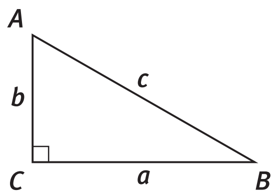
Write your answers on notebook paper.
Show your work.

Lesson 20-1

Write each expression in simplest radical form.

1. $\sqrt{40}$
2. $\sqrt{128}$
3. $\sqrt{162}$

Use the Pythagorean Theorem and the triangle below for Items 4 and 5. Recall that the Pythagorean Theorem states that for all right triangles, $a^2 + b^2 = c^2$.



4. In the right triangle, if $a = 3$ and $b = 6$, what is the value of c?
   A. $3\sqrt{5}$ **B.** 9
   C. $9\sqrt{5}$ **D.** 45
5. In the right triangle, if $a = 12$ and $b = 15$, what is the value of c?

Simplify each expression.

6. $\sqrt{4m^7}$
7. $3\sqrt[4]{16n^8}$
8. $\sqrt[3]{16x^4}$

Write each of the following as a radical expression.

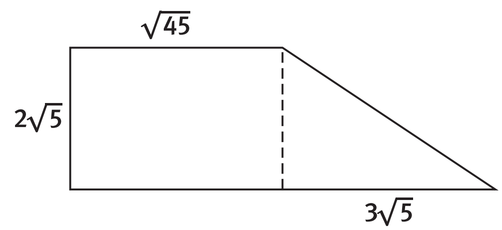
9. $15^{\frac{2}{5}}$
10. $(2p)^{\frac{1}{3}}$
11. $16x^{\frac{3}{4}}$
12. Which of the following expressions is **not** equivalent to $(8x)^{\frac{2}{3}}$?
    A. $4\sqrt[3]{x^2}$ **B.** $\sqrt[3]{8x^2}$
    C. $4x^{\frac{2}{3}}$ **D.** $8^{\frac{2}{3}}x^{\frac{2}{3}}$

Lesson 20-2

Write each expression in simplest radical form. State whether each result is rational or irrational.

13. $4\sqrt{27} + 6\sqrt{12}$
14. $8\sqrt{6} + 2\sqrt{12} + 5\sqrt{3} - \sqrt{54}$
15. $3\sqrt{36} - 5\sqrt{16} + 4$
16. Which of the following is the difference of $9\sqrt{20}$ and $2\sqrt{5}$?
    A. $7\sqrt{15}$ **B.** $16\sqrt{5}$
    C. $9\sqrt{5}$ **D.** $7\sqrt{5}$

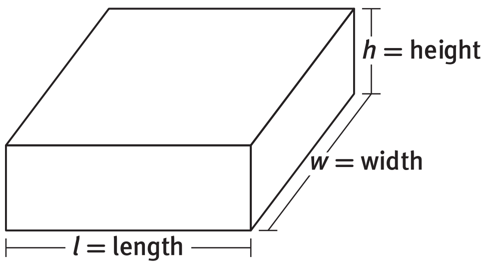
The figure below is composed of a rectangle and a right triangle. Use the figure for Items 17–19.



17. Determine the perimeter of the rectangle.
18. Determine the perimeter of the triangle.
19. Determine the perimeter of the composite figure.
20. A student was asked to completely simplify the expression $3\sqrt{3} + \sqrt{12} + 2\sqrt{3}$. The student wrote $5\sqrt{3} + \sqrt{12}$. Do you agree with the student's answer? Explain.

Lesson 20-3

The figure shows a rectangular prism. The volume of the rectangular prism is the product of the length, width, and height. Use the figure for Items 21–24.



21. If $l = \sqrt{3}$, $w = \sqrt{2}$, and $h = \sqrt{6}$, what is the volume of the rectangular prism? Is the volume rational or irrational?

22. If $l = 3\sqrt{3}$, $w = 2\sqrt{2}$, and $h = 5\sqrt{10}$, what is the volume of the rectangular prism? Is the volume rational or irrational?

23. If the volume of the rectangular prism is 20, the length is $\sqrt{3}$, and the width is $\sqrt{5}$, what is the height?

24. If the volume of the rectangular prism is $24\sqrt{3}$, the height is $2\sqrt{2}$, and the width is $3\sqrt{10}$, what is the length?

Write each expression in simplest form.

25. $\left(2\sqrt{2x^2}\right)\left(3x\sqrt{x^2}\right)$

26. $\left(6p\sqrt{p^3}\right)\left(0.2\sqrt{16p}\right)$

27. $\left(4\sqrt[3]{8m}\right)\left(7m\sqrt[3]{m^5}\right)$

28. $\dfrac{3\sqrt{32}}{\sqrt{2}}$

29. $\dfrac{\sqrt[3]{81x}}{\sqrt[3]{3}}$

30. $\dfrac{\sqrt{x^2y^5}}{\sqrt{y}}$

31. Which of the following expressions cannot be simplified any further?

A. $\sqrt{\dfrac{5}{2}}$ **B.** $\dfrac{\sqrt{5}}{\sqrt{2}}$

C. $\sqrt{52}$ **D.** $5\sqrt{2}$

32. Elena was asked to simplify the expression $\left(2\sqrt{12x}\right)\left(4x\sqrt{3}\right)$. Her answer was $48x\sqrt{x}$.

a. Explain how Elena can use her calculator to check whether her answer is reasonable.

b. Is Elena's answer correct? If not, explain Elena's mistake and give the correct answer.

The time, T, in seconds, it takes the pendulum of a clock to swing from one side to the other side is given by the formula $T = \pi\sqrt{\dfrac{l}{32}}$, where l is the length of the pendulum, in feet. The clock ticks each time the pendulum is at the extreme left or right point.

Use this information for Items 33–36.

33. If the pendulum is 4 feet long, how long does it take the pendulum to swing from left to right? Give an exact value in terms of π.

34. If the pendulum is 8 feet long, how long does it take the pendulum to swing from left to right? Give an exact value in terms of π.

35. If the pendulum is shortened, will the clock tick more or less often? Explain how you arrived at your conclusion.

36. Approximately what length of the pendulum will result in its swinging from one side to the other every second?

MATHEMATICAL PRACTICES
Construct Viable Arguments and Critique the Reasoning of Others

37. Amil knows that the formula for the area of a circle is $A = \pi r^2$. He says that the area of a circle with a radius of $2\sqrt{5}$ feet is $4\sqrt{5}\pi$ square feet. Is he correct? If not, describe his error.

Geometric Sequences

Go Viral!

Lesson 21-1 Identifying Geometric Sequences

My Notes

Learning Targets:

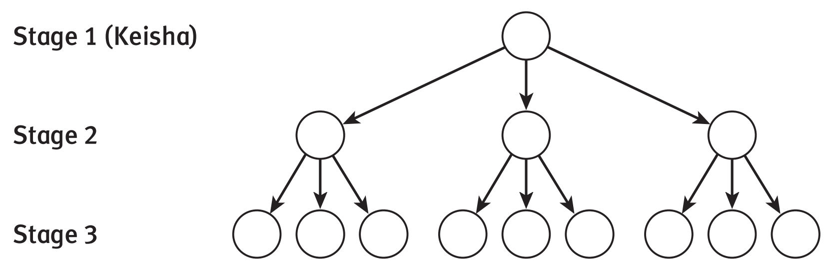
- Identify geometric sequences and the common ratio in a geometric sequence.
- Distinguish between arithmetic and geometric sequences.

SUGGESTED LEARNING STRATEGIES: Visualization, Look for a Pattern, Create Representations, Think-Pair-Share, Sharing and Responding

For her Electronic Communications class, Keisha has been tasked with investigating the effects of social media. She decides to post a video in cyberspace to see if she can make it go viral.

To get things started, Keisha e-mails the video link to three of her friends. In the message, she asks each of the recipients to forward the link to three of his or her friends. Whenever a recipient forwards the link, Keisha asks him or her to attach the following message: *After watching, please forward this video link to three of your friends who have not yet received it.*

One way to visually represent this situation is with a tree diagram. A **tree diagram** shows all the possible outcomes of an event.



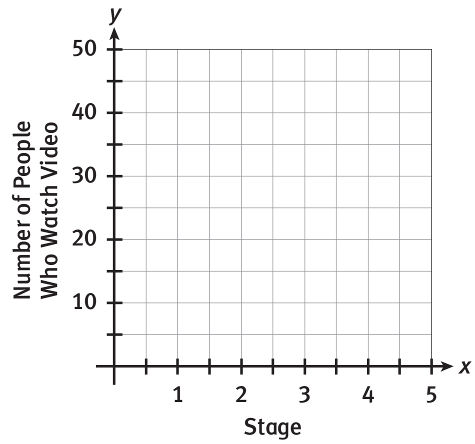
1. Use the tree diagram to help you complete the table. (Assume that everyone who receives the video link watches the video.)

Stage	Number of People Who Watch the Video
1	1
2	
3	
4	

2. **Express regularity in repeated reasoning.** Describe any patterns you notice in the table.

My Notes

3. Use the table of values to graph the viral video situation.



4. Is the relationship a linear relationship? Justify your response.

5. Is the graph the graph of a **function**? If so, what is the domain?

MATH TERMS

A **function** is a relation in which every input is paired with exactly one output.

MATH TERMS

geometric sequence
common ratio

The number of people who have received the video link at each stage form a *geometric sequence.* A **geometric sequence** is a sequence of values in which a nonzero constant ratio exists between consecutive terms. The constant ratio is called the **common ratio** and is typically denoted by the letter *r*. The common ratio is the value that each term is multiplied by to get the next term.

6. a. Write the numbers of people who have received the video as a sequence.

   b. **Reason quantitatively.** Identify the common ratio. Justify your response.

My Notes

7. Identify each sequence as arithmetic, geometric, or neither. If it is arithmetic, state the common difference. If it is geometric, state the common ratio.
   a. $5, 8, 11, 14, \ldots$

   b. $18, 6, 2, \frac{2}{3}, \ldots$

   c. $1, 4, 9, 16, \ldots$

   d. $-1, 4, -16, 64, \ldots$

   e. $16, -8, 4, -2, \ldots$

MATH TERMS

An **arithmetic sequence** is a sequence in which the difference between consecutive terms is constant. This difference is called the **common difference.**

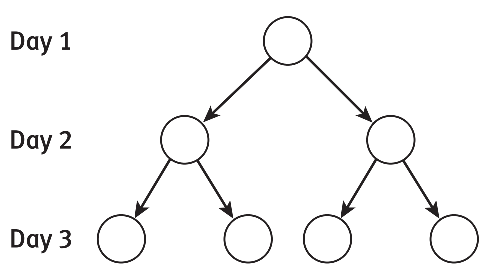
Check Your Understanding

Identify each sequence as arithmetic, geometric, or neither. If it is arithmetic, state the common difference. If it is geometric, state the common ratio.

8. $10, 8, 6, 4, \ldots$ **9.** $2, \frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \ldots$ **10.** $9, -3, 1 -\frac{1}{3}, \ldots$

LESSON 21-1 PRACTICE

A cell divides in two every day. The tree diagram shows the first few stages of this process. Use the tree diagram for Items 11–13.



11. Make a table of values to represent the scenario shown in the tree diagram.

12. Does the tree diagram represent a geometric sequence? If so, what is the common ratio?

13. If the diagram were extended to a sixth day, how many circles would there be on Day 6?

14. **Reason abstractly.** Can a geometric sequence ever have a term equal to 0? Explain.

My Notes

Learning Targets:

- Write a recursive formula for a geometric sequence.
- Write an explicit formula for a geometric sequence.
- Use a formula to find a given term of a geometric sequence.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Discussion Groups, Think-Pair-Share, Construct an Argument, Sharing and Responding

Remember that the numbers in a sequence are called *terms*, and you can use sequence notation a_n or function notation $f(n)$ to refer to the nth term.

1. a. Use the above notation to rewrite the first four terms of the viral video sequence. Also write the common ratio.

$a_1 = f(1) =$

$a_2 = f(2) =$

$a_3 = f(3) =$

$a_4 = f(4) =$

$r =$

b. What is the value of the term following $a_4 = f(4)$? Write an expression to represent this term using a_4 and the common ratio.

MATH TERMS

A **recursive formula** is a formula that gives the first term and a process to apply to each term to find the next term.

Just as with arithmetic sequences, you can use a **recursive formula** to represent a geometric sequence.

2. Complete the table below for the viral video sequence.

Term	Sequence Representation Using Common Ratio	Function Representation Using Common Ratio	Numerical Value (number of people who have seen the video)
$a_1 = f(1)$	1	1	1
$a_2 = f(2)$	$3(1) = 3a_1$	$3(1) = 3f(1)$	3
$a_3 = f(3)$	$3(3) = 3a_2$	$3(3) = 3f(2)$	
$a_4 = f(4)$			
$a_5 = f(5)$			
. . .	. . .	. . .	. . .
$a_n = f(n)$			—

3. The recursive formulas for the viral video sequence are partially given below. Complete the formulas by writing the expressions for a_n and $f(n)$.

$$\begin{cases} a_1 = 1 \\ a_n = \end{cases} \qquad \begin{cases} f(1) = 1 \\ f(n) = \end{cases}$$

My Notes

Check Your Understanding

Write a recursive formula for each geometric sequence. Include the recursive formula in function notation.

4. 64, 16, 4, 1, . . .

5. 64, −16, 4, −1, . . .

6. −1, 1, −1, 1, . . .

7. Use the recursive formula to find a_6, a_7, and a_8 for the viral video sequence. Explain your results.

8. Why might it be difficult to find the 100th term of the viral video sequence using the recursive formula?

As with arithmetic sequences, geometric sequences can be represented with explicit formulas. The terms in a geometric sequence can be written as the product of the first term and a power of the common ratio.

9. For the viral video sequence, identify a_1 and r. Then fill in the missing exponents and blanks.

$a_1 =$ ________ $r =$ ________

$a_2 = 1 \cdot 3^{\square} = 3$

$a_3 = 1 \cdot 3^{\square} = 9$

$a_4 = 1 \cdot 3^{\square} =$ ________

$a_5 = 1 \cdot 3^{\square} =$ ________

$a_6 = 1 \cdot 3^{\square} =$ ________

$a_{10} = 1 \cdot 3^{\square} =$ ________

10. Express regularity in repeated reasoning. Describe any patterns you observe in your responses to Item 9. Then use a_1, r, and n to write a formula for the nth term of any geometric sequence.

11. Write the explicit formula for the viral video sequence. Use the formula to determine the 12th term of the sequence. What does the 12th term represent?

My Notes

12. The explicit formula for a geometric sequence can be thought of as a function.

a. What is the input? What is the output?

b. State the domain of the function.

c. Rewrite the explicit formula for the viral video sequence using function notation.

d. **Use appropriate tools strategically.** Use a graphing calculator to graph your function from Part (c). Is the function linear or nonlinear? Justify your response.

13. Consider the geometric sequence 5, 10, 20, 40, . . .

a. Write the explicit formula for the sequence.

b. How can you check that your formula is correct?

c. Determine the 16th term in the sequence.

d. Use function notation to write the explicit formula for the sequence.

e. What is the value of $f(10)$? What does it represent?

14. **a.** Write the explicit formula for the geometric sequence 32, 16, 8, 4, . . .

b. Determine the 9th term in the sequence.

My Notes

15. The explicit formula for a geometric sequence is $a_n = 6 \cdot 3^{n-1}$. State the recursive formula for the sequence. Include the recursive formula in function notation.

Check Your Understanding

16. How can you use the recursive formula for a geometric sequence to write the explicit formula?

Write the explicit formula for each geometric sequence. Then determine the 6th term of each sequence.

17. 1, 5, 25, . . .

18. 48, −24, 12, . . .

19. $\begin{cases} a_1 = -81 \\ a_n = -\frac{1}{3}a_{n-1} \end{cases}$

20. **Make sense of problems.** Revisit the viral video scenario at the beginning of the activity. How many stages will it take until 1 million new people receive the link to the viral video? Explain how you found your answer.

Check Your Understanding

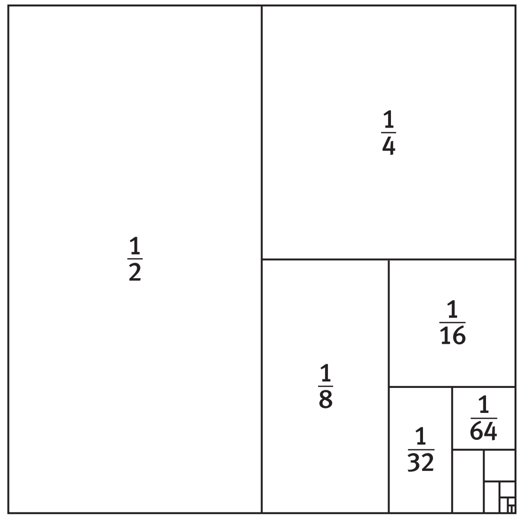
21. Write a recursive formula for the geometric sequence whose explicit formula is $a_n = 1 \cdot (-2)^{n-1}$. Include the recursive formula in function notation.

22. Write an explicit formula for the sequence $\begin{cases} a_1 = 3 \\ a_n = 2a_{n-1} \end{cases}$.

My Notes

LESSON 21-2 PRACTICE

The diagram below shows a square repeatedly divided in half. The entire square has an area of 1 square unit. The number in each region is the area of the region. Use the diagram for Items 23–25.



23. Write a geometric sequence to describe the areas of successive regions.

24. Write an explicit formula for the geometric sequence that you wrote in Item 23.

25. **Model with mathematics.** What is the 10th term of the sequence? What does it represent?

26. The explicit formula for a geometric sequence is $f(n) = 5(-2)^{n-1}$. Give the recursive formula for the sequence.

ACTIVITY 21 PRACTICE

**Write your answers on notebook paper.
Show your work.**

Lesson 21-1

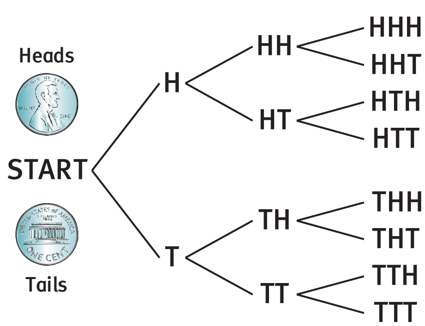
In Items 1 and 2, assume that the first term of a sequence is -3.

1. Write the first four terms of the sequence if it is an arithmetic sequence with common difference $-\frac{1}{3}$.

2. Write the first four terms of the sequence if it is a geometric sequence with common ratio $-\frac{1}{3}$.

The tree diagram below shows the number of possible outcomes when tossing a coin a number of times. For example, if you toss a coin once (Stage 1), there are two possible outcomes: heads (H) and tails (T). If you toss a coin twice (Stage 2), there are four possible outcomes for the two tosses: HH, HT, TH, and TT.

Use the tree diagram for Items 3–5.



3. How many possible outcomes are there when you toss a coin 4 times?

4. Identify the common ratio of the sequence represented by the tree diagram.

5. How many possible outcomes are there when you toss a coin 23 times? Express your answer using exponents.

For Items 6–10, identify each sequence as arithmetic, geometric, or neither. If it is arithmetic, state the common difference. If it is geometric, state the common ratio.

6. 17, 25, 33, 41, . . .

7. 1, 3, 6, 10, 15, . . .

8. $-27, -9, -3, -1, \ldots$

9. 0.1, 0.5, 0.9, 1.3, . . .

10. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

11. A geometric sequence begins with the value 1 and has a common ratio of -2. Identify the eighth term in the sequence.
A. -128 **B.** 128
C. -256 **D.** 256

12. A geometric sequence begins with the value 2 and has a common ratio of $\frac{1}{2}$. Identify the fifth term in the sequence.
A. 4 **B.** $\frac{1}{4}$
C. $\frac{1}{8}$ **D.** $\frac{1}{16}$

13. Which of the following is a *false* statement about the sequence 2, 4, 8, 16, 32, . . . ?
A. The common ratio of the sequence is 2.
B. The tenth term of the sequence is 2^{10}.
C. Every term of the sequence is even.
D. The number 216 appears in the sequence.

14. Give an example of a geometric sequence with a common ratio of 0.2. Write at least the first four terms of the sequence.

Lesson 21-2

Write a recursive formula for each geometric sequence. Include the recursive formula in sequence notation.

15. 7, 21, 63, 189, . . .

16. 100, 10, 1, 0.1, . . .

17. −10, 20, −40, 80, . . .

Write the explicit formula for each geometric sequence. Then determine the 8th term of each sequence.

18. 4, 16, 64, 256, . . .

19. $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \ldots$

20. $\begin{cases} a_1 = 5 \\ a_n = -3a_{n-1} \end{cases}$

A contestant on a game show wins $100 for answering a question correctly in Round 1. In each subsequent round, the contestant's winnings are doubled if she gives a correct answer. If the contestant gives an incorrect answer, she loses everything. Use this information for Items 21–23.

21. Write an explicit formula that gives the contestant's winnings in round n, assuming she answers all questions correctly.

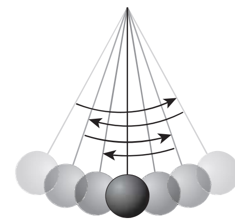
22. How much does the contestant win in Round 10, assuming she answers all questions correctly?

23. How many rounds does a contestant need to play in order to answer a question worth at least $1,000,000?

24. A geometric sequence is given by the recursive formula $\begin{cases} f(1) = -6 \\ f(n) = -\frac{1}{2} f(n-1) \end{cases}$. Which of the following is a term in the sequence?

A. $\frac{3}{4}$
B. $-\frac{3}{4}$
C. $\frac{3}{2}$
D. -3

Each time a pendulum swings, the distance it travels decreases, as shown in the figure.



Pendulum Swing

The table shows how far the pendulum travels with each swing. Use this table for Items 25–27.

Swing Number	Distance Traveled (cm)
1	80
2	60
3	45
4	33.75

25. Write the explicit formula for the pendulum situation.

26. How far will the pendulum travel on the seventh swing?

27. How many swings will it take for the pendulum to travel less than 10 cm?

The game commission observes the fish population in a stream and notices that the number of trout increases by a factor of 1.5 every week. The commission initially observed 80 trout in the stream. Use this information for Items 28–31.

28. Write the explicit formula for the trout situation.

29. Make a graph of the population growth.

30. If this pattern continues, how many trout will be in the stream on the fifth week?

31. If this pattern continues, on what week will the trout population exceed 500?

MATHEMATICAL PRACTICES
Construct Viable Arguments and Critique the Reasoning of Others

32. Samir says that it is possible for a sequence to be both an arithmetic sequence and a geometric sequence. Do you agree or disagree? Explain.

Exponential Functions

ACTIVITY 22

Protecting Your Investment

Lesson 22-1 Exponential Functions and Exponential Growth

My Notes

Learning Targets:

- Understand the definition of an exponential function.
- Graph and analyze exponential growth functions.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Create Representations, Look for a Pattern, Interactive Word Wall, Predict and Confirm, Think-Pair-Share

The National Association of Realtors estimates that, on average, the price of a house doubles every ten years. Tony's grandparents bought a house in 1960 for $10,000. Assume that the trend identified by the National Association of Realtors applies to Tony's grandparents' house.

1. What was the value of Tony's grandparents' house in 1970 and in 1980?

2. Compute the difference in value from 1960 to 1970.

3. Compute the ratio of the 1970 value to the 1960 value.

MATH TIP

The ratio of the quantity a to the quantity b is evaluated by dividing a by b (ratio of a to $b = \frac{a}{b}$).

4. Complete the table of values for the years 1960 to 2010.

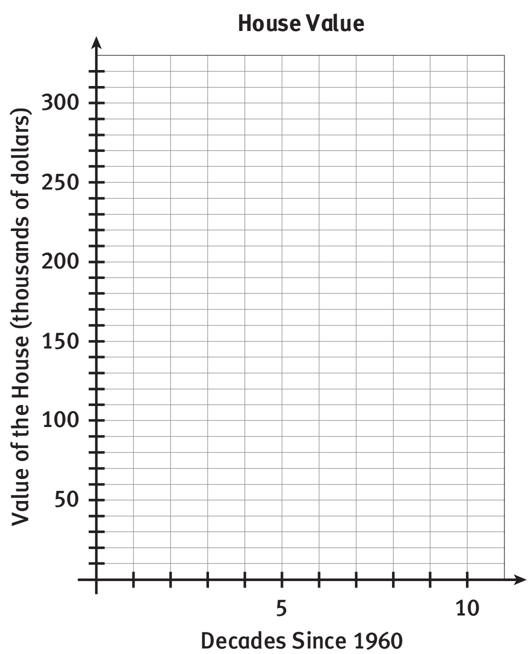
House Value				
Year	Decades Since 1960	Value of House	Difference Between Values of Consecutive Decades	Ratio of Values of Consecutive Decades
1960	0	$10,000	—	—
1970				
1980				
1990				
2000				
2010				

5. What patterns do you recognize in the table?

My Notes

6. Write the house values as a sequence. Identify the sequence as arithmetic or geometric and justify your answer.

7. Using the data from the table, graph the ordered pairs (decades since 1960, house value) on the coordinate grid below.



8. The data comparing the number of decades since 1960 and value of the house are not linear. Explain why using the table and the graph.

9. **Make use of structure.** Using the information that you have regarding the house value, predict the value of the house in the year 2020. Explain how you made your prediction.

10. Tony would like to know what the value of the house was in 2005. Using the same data, predict the house value in 2005. Explain how you made your prediction.

The increase in house value for Tony's grandparents' house is an example of **exponential growth**. Exponential growth can be modeled using an **exponential function**.

> **Exponential Function**
> A function of the form $f(x) = a \cdot b^x$, where x is the domain value, $f(x)$ is the range value, $a \neq 0$, $b > 0$, and $b \neq 1$.

Lesson 22-1
Exponential Functions and Exponential Growth

My Notes

In exponential growth, a quantity is multiplied by a constant factor greater than 1 during each time period.

11. The value of Tony's grandparents' house is growing exponentially because it is multiplied by a constant factor for each decade. What is this constant factor?

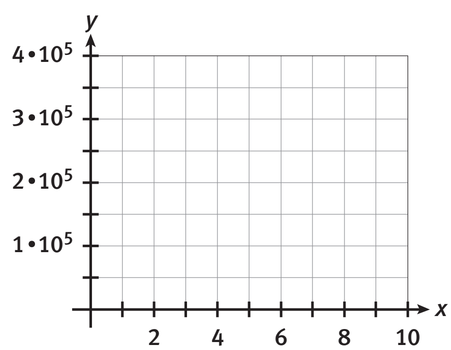
A function that can be used to model the house value is $h(t) = 10{,}000 \cdot (2)^t$. Use this function for Items 12–17.

12. Identify the meaning of $h(t)$ and t. What are the reasonable domain and range?

13. Describe how your answer to Item 11 is related to the function $h(t) = 10{,}000 \cdot (2)^t$.

14. Complete the table of values for t and $h(t)$. Then graph the function $h(t)$ on the grid below.

t	$h(t)$



TECHNOLOGY TIP

Graph $h(t)$ on a graphing calculator. Find the y-coordinate when x is about 4.5. The value should be close to your calculated value in Item 17.

15. What was the value of the house in 1960? Describe how this value is related to the function $h(t) = 10{,}000 \cdot (2)^t$ and to the graph.

16. Calculate the value of the house in the year 2020. How does the value compare with your prediction in Item 9?

17. Calculate the value of the house in the year 2005. How does the value compare with your prediction in Item 10?

My Notes

Check Your Understanding

x	$g(x)$
0	
1	
2	
3	
4	

18. Copy and complete the table for the exponential function $g(x) = 3^x$.

19. Use your table to make a graph of $g(x)$.

20. Identify the constant factor for this exponential function.

Isaac evaluates the function modeling Tony's grandparents' house value, $h(t) = 10{,}000 \cdot (2)^t$, at $t = 2.5$. The variable t represents the number of decades since 1960.

21. What is $h(2.5)$?

22. For which year is Isaac estimating the house's value?

LESSON 22-1 PRACTICE

The value of houses in different locations can grow at different rates. The table below shows the value of Maddie's house from 1960 until 2010. Use the table for Items 23–25.

Year	Decades Since 1960	Value of House
1960	0	\$10,000
1970	1	\$15,000
1980	2	\$22,500
1990	3	\$33,750
2000	4	\$50,625
2010	5	\$75,938

23. Create a graph showing the value of Maddie's house from 1960 until 2010.

24. Explain how you know that the value of Maddie's house is growing exponentially.

25. What was the approximate value of Maddie's house in 1995?

The function $f(t) = 20{,}000 \cdot (1.2)^t$ can be used to find the value of Eduardo's house between 1970 and 2010, where the initial value of the function is the value of Eduardo's house in 1970.

26. Model with mathematics. Describe what the domain and range of the function mean in the context of Eduardo's house value.

27. What was the value of Eduardo's house in 1970?

28. Approximately how much was the house worth in 2000?

My Notes

Learning Targets:

- Describe characteristics of exponential decay functions.
- Graph and analyze exponential decay functions.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Create Representations, Predict and Confirm, Discussion Groups, Visualization

Radon, a naturally occurring radioactive gas, was identified as a health hazard in some homes in the mid 1980s. Since radon is colorless and odorless, it is important to be aware of the concentration of the gas. Radon has a *half-life* of approximately four days.

Tony's grandparents' house was discovered to have a radon concentration of 400 pCi/L. Renee, a chemist, isolated and eliminated the source of the gas. She then wanted to know the quantity of radon in the house in the days following so that she could determine when the house would be safe.

1. **Make sense of problems.** What is the amount of the radon in the house four days after the source was eliminated? Explain your reasoning.

2. Compute the difference in the amount of radon from Day 0 to Day 4.

3. Determine the ratio of the amount of radon on Day 4 to the amount of radon on Day 0.

CONNECT TO SCIENCE

All radioactive elements have a *half-life*. A half-life is the amount of time in which a radioactive element decays to half of its original quantity.

CONNECT TO SCIENCE

The US Environmental Protection Agency (EPA) recommends that the level of radon be below 4 pCi/L (picoCuries per liter) in any home. The EPA recommends that all homes be tested for radon.

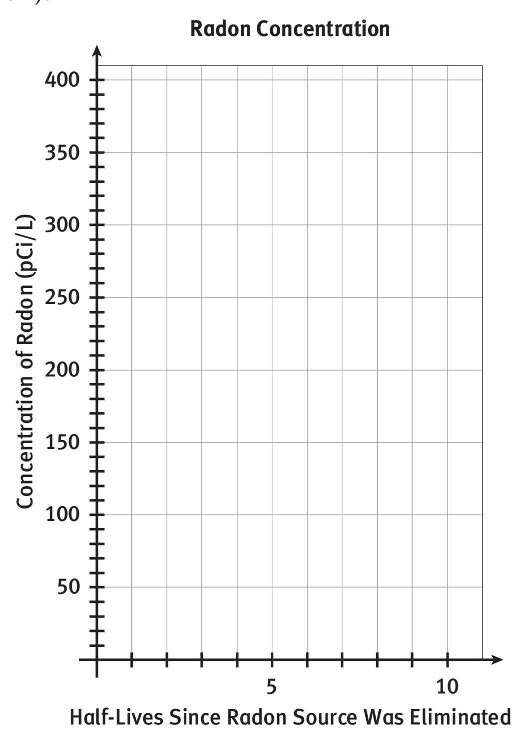
My Notes

4. Complete the table for the radon concentration.

Radon Concentration				
Half-Lives	**Days After Radon Source Was Eliminated**	**Concentration of Radon in pCi/L**	**Difference Between Concentration of Consecutive Half-Lives**	**Ratio of Concentrations of Consecutive Half-Lives**
0	0	400	—	—
1				
2				
3				

5. Express regularity in repeated reasoning. What patterns do you recognize in the table?

6. Graph the data in the table as ordered pairs in the form (half-lives, concentration).



7. The data that compares the number of half-lives and the concentration of radon are not linear. Explain why using the table of values and the graph.

8. Renee needs to know the concentration of radon in the house after 20 days. How many radon half-lives are in 20 days? What is the concentration after 20 days?

9. How many radon half-lives are in 22 days? Predict the concentration after 22 days.

The decrease in radon concentration in Tony's grandparents' house is an example of **exponential decay**. Exponential decay can be modeled using an exponential function.

In exponential decay, a quantity is multiplied by a constant factor that is greater than 0 but less than 1 during each time period.

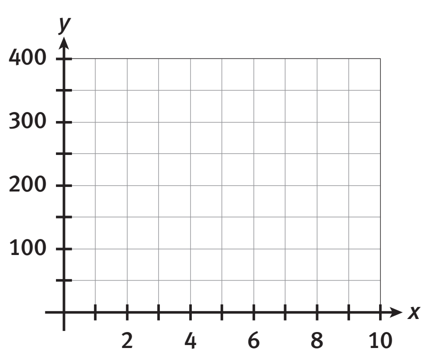
10. The concentration of radon is multiplied by a constant factor for each half-life. What is this constant factor?

A function that can be used to model the radon concentration is $r(t) = 400 \cdot \left(\frac{1}{2}\right)^t$. Use the function for Items 11–15.

11. Identify the meaning of $r(t)$ and t. What are the reasonable domain and range?

12. Describe how your answer to Item 10 is related to the function $r(t) = 400 \cdot \left(\frac{1}{2}\right)^t$.

13. Graph the function $r(t)$.



14. Describe how the original concentration of radon is related to the function $r(t) = 400 \cdot \left(\frac{1}{2}\right)^t$ and to the graph.

My Notes

15. Use the function to identify the concentration of radon after 20 days. How does the concentration compare with your prediction in Item 8?

16. Use the function to calculate the concentration of radon after 22 days. How does the concentration compare with your prediction in Item 9?

17. **Construct viable arguments.** Will the concentration of radon ever be 0? Explain your reasoning.

CONNECT TO AP

In calculus, you will discover what happens as functions approach 0.

Check Your Understanding

18. Copy and complete the table for the exponential function $g(x) = \left(\frac{1}{4}\right)^x$.

19. Identify the constant factor for this exponential function.

x	$g(x)$
0	
1	
2	
3	

LESSON 22-2 PRACTICE

20. The amount of medication in a patient's bloodstream decreases exponentially from the time the medication is administered. For a particular medication, a function that gives the amount of medication in a patient's bloodstream t hours after taking a 100 mg dose is $A(t) = 100\left(\frac{7}{10}\right)^t$. Use this function to find the amount of medication remaining after 2 hours.

21. Make a table of values and graph each function.

a. $h(x) = 2^x$

b. $l(x) = 3^x$

c. $m(x) = \left(\frac{1}{2}\right)^x$

d. $p(x) = \left(\frac{1}{3}\right)^x$

22. Which of the functions in Item 21 represent exponential growth? Explain using your table of values and graph.

23. Which of the functions in Item 21 represent exponential decay? Explain using your table of values and graph.

24. **Reason abstractly.** How can you identify which of the functions represent growth or decay by looking at the function?