Compositions and Congruence

ACTIVITY 10

More Rigid Motions

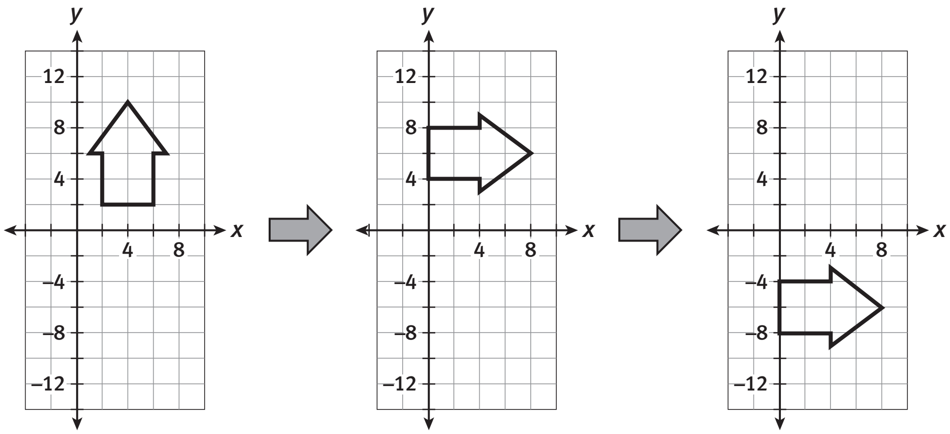
Lesson 10-1 Compositions of Transformations

My Notes

Learning Targets:

- Find the image of a figure under a composition of rigid motions.
- Find the pre-image of a figure under a composition of rigid motions.

SUGGESTED LEARNING STRATEGIES: Close Reading, Think-Pair-Share, Predict and Confirm, Self Revision/Peer Revision



Consider the series of transformations shown in the figures above. The first is $R_{(6,4),90^\circ}$, or a clockwise rotation of 90° about the point (4, 6). The second transformation is $r_{y=0}$, or a reflection across the x-axis. Together they are a ***composition of transformations***, which is two or more transformations performed in sequence.

The notation for a composition of transformations is similar to the way you express the composition of functions. The composition pictured above is described as $r_{y=0}\,(R_{(6,4),90^\circ})$. If a reflection across the x-axis were added as the third transformation of the sequence, then the notation would be $r_{x=0}(r_{y=0}(R_{(6,4),90^\circ}))$. Notice that the transformation that occurs first in the series is in the interior of the notation, and subsequent transformations are written outside of it.

1. **Attend to precision.** Write the notations for these compositions of transformations. Use the points $A(0, 0)$, $B(1, 1)$, and $C(0, -1)$.
   a. a clockwise rotation of 60° about the origin, followed by a translation by directed line segment $\overline{AB}$

   b. a reflection about the line $x = 1$, followed by a reflection about the line $x = 2$

   c. three translations, each of directed line segment $\overline{AC}$

MATH TERMS

A **composition of transformations** is a series of two or more transformations performed on a figure one after another.

DISCUSSION GROUP TIP

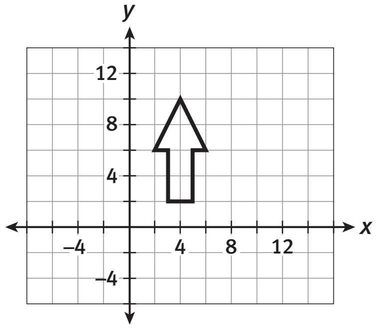
As needed, refer to the Glossary to review meanings of key terms. Incorporate your understanding into group discussions to confirm your knowledge and use of key mathematical language.

My Notes

MATH TIP

Use cut-out shapes to model each transformation in the composition. Remember that the notation shows the transformations in order from *right to left*.

The figure shows an arrow that points from a short horizontal line through (4, 2) to its tip at (4, 10).



2. Draw the image of the arrow mapped by the composition $r_{(x=4)}(R_{(4,0),90°})$. Label it A.

3. Draw the image of the arrow mapped by a composition of the same transformations in the reverse order: $R_{(4,0),90°}(r_{(x=4)})$. Label it B.

4. What single transformation maps the arrow to image A? Write the name of the transformation and its symbolic representation.

5. What single transformation maps the arrow to image B?

As demonstrated by Items 2 and 3, the order of transformations in a composition can affect the position and orientation of the image. And as shown by Items 4 and 5, a composition can produce the same image as a single translation, reflection, or rotation.

6. For each of these compositions, predict the single transformation that produces the same image.

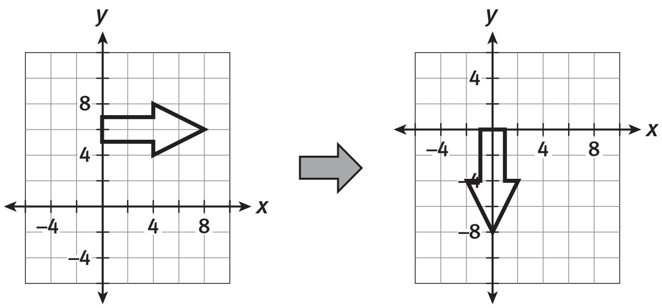
a. $T_{(1,1)}(T_{(0,1)}(T_{(1,0)}))$

b. $R_{O,90°}\,(R_{O,90°})$

c. $r_{(x=0)}(r_{(y=0)})$

My Notes

Use the figure for Items 7–9.



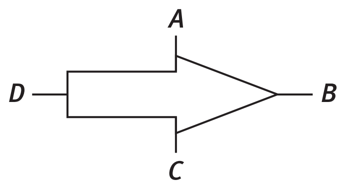
7. Identify a composition of transformations that could map the arrow on the left to the image of the arrow on the right.

8. Consider the composition you identified in Item 7 but with the transformations in reverse order. Does it still map the arrow to the same image?

9. Identify a composition that undoes the mapping, meaning it maps the image of the arrow on the right to the pre-image on the left.

You can also find combinations of transformations away from the coordinate plane.

10. Points A, B, C, and D are points on the right-pointing arrow shown here. Predict the direction of the arrow after it is mapped by these combinations.



a. $T_{\overline{DB}}(T_{\overline{AC}})$

b. $r_{\overline{D'B'}}(R_{D,90^\circ})$

c. $R_{A,180^\circ}(r_{\overline{AC}})$

My Notes

MATH TIP

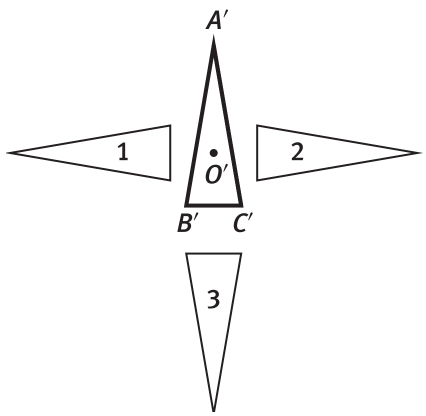
You can think of an inverse transformation as a "reversing" or "undoing" of the transformation. It maps the image to the pre-image.

Like many functions, transformations have inverses, which are transformations that map the image back to the pre-image. The inverse of transformation T is designated T^{-1}, and it has the property $T^{-1}(T(P)) = P$ for all points P. The table lists the general formulas for the inverses of translations, rotations, and reflections.

Function	Notation	Inverse
Translation by directed line segment $\overline{AB}$	$T_{\overline{AB}}$	$(T_{\overline{AB}})^{-1} = T_{\overline{BA}}$
Reflection about line l	r_l	$(r_l)^{-1} = r_l$
Rotation of m degrees about point O	R_{O,m°	$(R_{O,m^\circ})^{-1} = R_{O,-m^\circ}$

Example A

Isosceles triangle $A'B'C'$ is shown in the diagram. It is the image of the combination $T_{\overline{BC}}(R_{O,90^\circ})$, in which point O is the center of the triangle.



Is the pre-image shown by triangle 1, 2, or 3?

Step 1: Find the inverse combination.

$(R_{O,90^\circ})^{-1} = (R_{O',-90^\circ})$, and $(T_{\overline{BC}})^{-1} = T_{\overline{C'B'}}$

So, the inverse combination is $R_{O,-90^\circ}(T_{\overline{CB}})$.

Step 2: Determine the result of the translation and rotation.

When the pre-image is translated by the directed line segment $\overline{C'B'}$, and then rotated 90° counterclockwise about its center, the result is triangle 1.

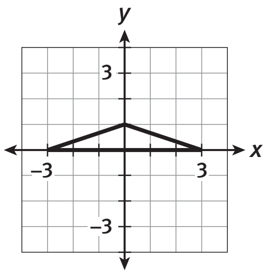
Try These A

a. Find a composition that maps triangle 2 to triangle $A'B'C'$.
b. Find a composition that maps triangle 3 to triangle $A'B'C'$.

My Notes

Check Your Understanding

An isosceles triangle has vertices at $(-3, 0)$, $(0, 1)$, and $(3, 0)$.



11. Draw the image of the triangle after the combination $T_{(0,2)}(R_{O,180^\circ})$.

12. Identify the inverse transformation.

13. Compare the mapping of the triangle produced by $T_{(0,2)}(R_{O,180^\circ})$ with the mapping produced by $r_{(y=1)}$.

LESSON 10-1 PRACTICE

14. Construct viable arguments. Give examples of a combination of rotation R_{O,m° $(0 < m < 360)$ and transformation T that is commutative (i.e., for all points P, $T(R_{O,m^\circ}(P)) = R_{O,m^\circ}(T(P))$ and that is not commutative (i.e., for at least one point P, $T(R_{O,m^\circ}(P)) \neq R_{O,m^\circ}(T(P))$).

15. The tail of an arrow is placed at $(0, 0)$ and its tip at $(3, 0)$, and it serves as the pre-image for four compositions. Complete the table to show the compositions and images.

Composition	Image (position of tip, direction of arrow)
$T_{(3,-3)}\,R_{0,90^\circ}$	
$r_{(x=3)}(R_{(3,0),180^\circ})$	
	(0, 6); left
	(−3, 0); down

MATH TIP

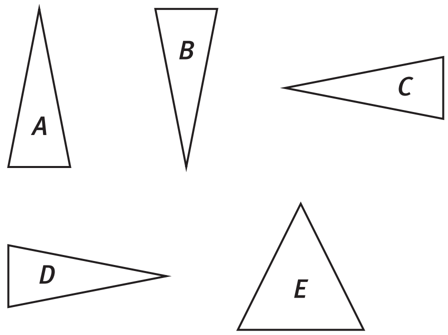
If a transformation is commutative, the order in which the transformations are performed does not matter—the resulting image will be the same.

My Notes

Learning Targets:

- Determine whether given figures are congruent.
- Specify a sequence of rigid motions that will carry a given figure to a congruent figure.

SUGGESTED LEARNING STRATEGIES: Visualization, Close Reading, Think-Pair-Share, Predict and Confirm



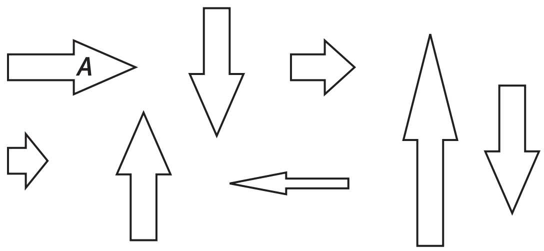
MATH TERMS

Two figures are **congruent** if and only if a composition of rigid motions maps one to the other.

Consider the five labeled triangles shown here. Isosceles triangles A, B, C, and D are ***congruent*** because a series of rigid motions—translations, rotations, or reflections—can map any of the triangles onto another. The series could involve a rotation of 90° or 180°, followed by a translation. However, no such combination can map either of the four triangles onto equilateral triangle E, nor does the inverse combination exist that would map triangle E onto triangle A, B, C, or D. Therefore, triangle E is not congruent to the other triangles.

The definition of congruent figures applies both on and off the coordinate plane. It also agrees with an intuitive understanding of the concept of congruency. For example, if you trace a figure, then rotate or reflect the tracing, the tracing will fit exactly over any congruent figures and will not fit over noncongruent figures.

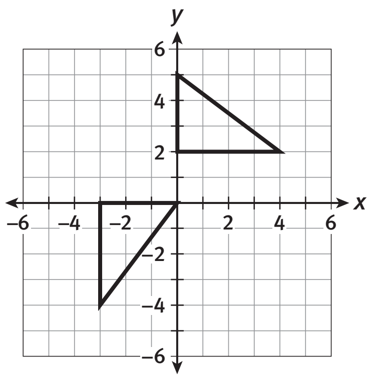
1. Shade all of the figures that appear to be congruent to figure A.



My Notes

To prove that two figures are congruent, a specific combination of rigid motions must be found that maps each point P of one figure to corresponding point P' of the other figure.

2. **Make use of structure.** Predict whether the two triangles shown in the figure are congruent. Explain your prediction.



3. How can your prediction be confirmed?

4. Could a single translation, reflection, or rotation map one of the triangles onto the other? Explain.

5. How could a rotation followed by a translation map one triangle onto another?

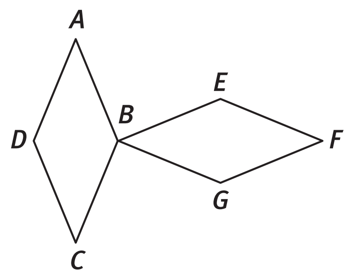
6. Propose a composition of rotation and translation to map the triangle in the first quadrant onto the triangle in the third quadrant. Then complete the table to test your proposal.

Pre-image:	Rotation: ________	Translation: ________
(0, 2)		(–3, 0)
(0, 5)		(0, 0)
(4, 0)		(–5, –4)

7. Was your prediction correct? Explain.

My Notes

Use the figure for Items 8–10.



8. Predict whether quadrilaterals *ABCD* and *BEFG* are congruent.

9. How could you test your prediction?

10. Follow your plan to test your prediction. Identify the rigid motions that you applied.

MATH TIP

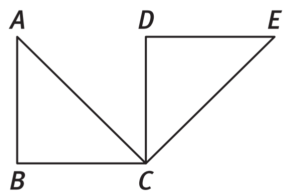
There may be many compositions of rigid motions that map one figure onto another. Only one composition is necessary to show congruence.

Check Your Understanding

11. An arrow is placed with its base at the origin and its tip at the point (0, 2). For each arrow listed below, find the rigid motion or composition that shows the two arrows are congruent.

a. base at (3, 3), tip at (3, 5)
b. base at (−2, 2), tip at (−2, 0)
c. base at (4, 0), tip at (6, 0)

12. Consider the right triangles shown below.



a. Identify a composition of reflections that shows that these two triangles are congruent.
b. Identify a rotation that shows that these two triangles are congruent.

My Notes

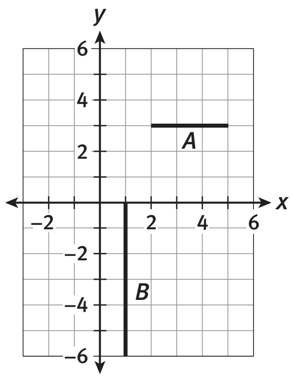
MATH TIP

Two triangles are congruent by SAS (side-angle-side).

In Activity 4, you learned another definition of congruent figures. This definition involves the measurements of sides and angles of a figure. For example, if two triangles each have sides of lengths a, b, and c, then the triangles are congruent. Congruency can also be shown when two triangles each have two sides of lengths a and b, and the angle between the two sides is the same in both triangles.

If two figures are congruent by one of the two definitions, are they congruent by the other definition as well? As you will see, the answer is yes.

Line A connects points (2, 3) and (5, 3). Line B connects points (1, 0) and (1, −6).



13. Find a composition of two rigid motions that maps line B to a line B' that overlaps line A.

14. What pair of points does line B' connect?

15. Can you add a third rigid motion to the composition that maps B'? Explain why or why not.

16. In the composition you identified, what subsection of line B maps onto line A exactly? What is the length of this subsection?

My Notes

As you've seen in this lesson, two congruent figures have corresponding sides of equal length and corresponding angles of equal measure. Congruence also has the following properties:

- reflexive (Every figure is congruent to itself.)
- symmetric (If A is congruent to B, then B is congruent to A.)
- transitive (If A is congruent to B and B is congruent to C, then A is congruent to C.)

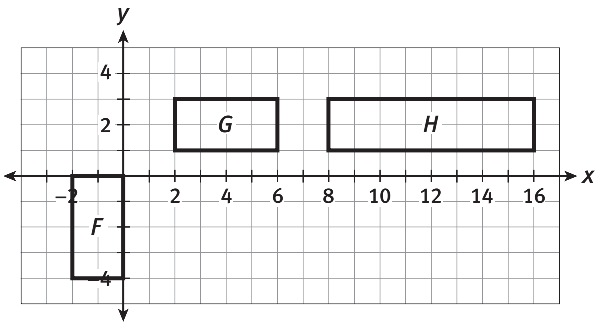
Check Your Understanding

17. Explain why a circle of radius 1 is not congruent to a circle of radius 2.

18. Figure A is transformed by a series of n rigid motions, in which n is a positive integer. Explain why figure A is congruent to its image after this series.

LESSON 10-2 PRACTICE

Rectangles F, G, and H are positioned on the coordinate plane as shown in the figure. Use the figure for Items 19–21.



19. Explain why figure F is congruent to figure G.

20. Explain why figure G is not congruent to figure H.

21. Use the answers to Items 19 and 20 to explain why figure F is not congruent to figure H.

22. **Make sense of problems.** The endpoints of $\overline{AB}$ are $A(-2, -1)$ and $B(1, 3)$. Suppose you know that $\overline{CD} \cong \overline{AB}$ and that the coordinates of C are $C(1, 0)$. What are possible coordinates for point D? Is there more than one answer? Explain.

ACTIVITY 10 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 10-1

1. Write the notations for these compositions of transformations.

a. a translation of directed line segment from the origin to (0, −3), followed by a reflection around the line $y = x$

b. a reflection about the line $y = 1$, followed by a clockwise rotation of 90 degrees about the origin

2. An arrow is placed with its base at (1, 1) and its tip at (1, 5). Identify the positions of the base and tip of the images of these compositions.

a. $r_{(x=0)}(R_{(1,0),90^\circ})$

b. $T_{(1,1)}((R_{O,-90^\circ})T_{(-1,-1)})$

c. $R_{(3,3),-90^\circ}(R_{(1,3),90^\circ})$

3. For any given figure A, does $r_{(x=0)}(R_{(2,0),90^\circ}(A))$ equal $R_{(2,0),90^\circ}(r_{(x=0)}(A))$? Use an example to explain your answer.

4. Identify the inverses of these transformations and compositions.

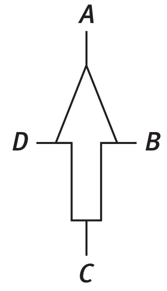
a. $r_{(y=0)}$

b. $T_{(1,3)}$

c. $R_{O,-90^\circ}(T_{(0,-1)})$

d. $r_{x=2}(T_{(1,2)}(r_{x=0}))$

5. An up-pointing arrow is defined by points A, B, C, and D, as shown below. Identify the direction of the arrow after these compositions.



a. $r_{\overline{BD}}\left(r_{\overline{AC}}\right)$

b. $T_{\overline{BD}}\left(R_{A,-90^\circ}\right)$

c. $R_{D,90^\circ}\left(R_{A,180^\circ}\right)$

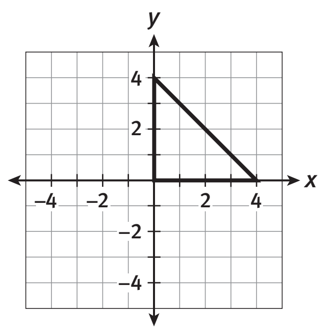
6. For each of these compositions, identify the single rigid motion that performs the same mapping.

a. $T_{(1,-6)}\,(T_{(-1,8)})$

b. $r_{(x=0)}\,(r_{(y=0)})$

c. $R_{(2,0),90^\circ}\left(R_{(2,0),180^\circ}\right)$

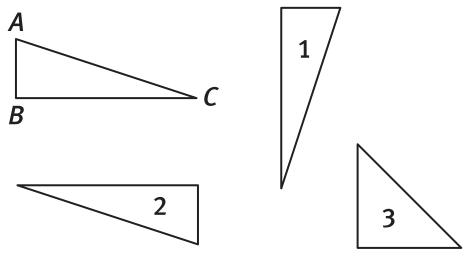
7. A right isosceles triangle is placed with vertices at (0, 0), (4, 0), and (0, 4).



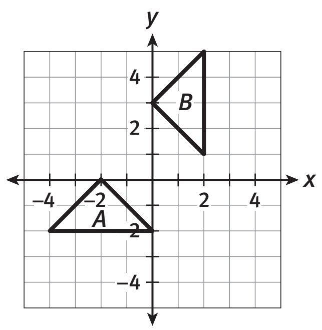
Compare the images of the triangle produced by $r_{y=0}(R_{O,180^\circ})$, $r_{y=x}(R_{O,90^\circ})$ and by $r_{x=0}(r_{y=x})$.
Hint: For any given point P, exactly two of the combinations map P to the same point.

Lesson 10-2

8. Identify the congruent triangles to triangle *ABC*. For the congruent triangles, identify the specific rotation or reflection that, along with a translation, would show congruence.



9. Which combination shows that triangles *A* and *B* are congruent?



A. $T_{(2,3)}(R_{(-2,0),-90°}(A))$

B. $R_{(O,90°)}(r_{(y=1.5)}(A))$

C. $r_{(x=0)}(r_{(y=1.5)}(A))$

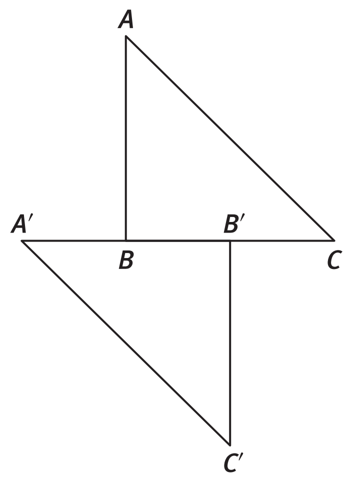
D. $T_{(0,5)}(R_{(0,-2),-60°}(A))$

10. Arrow *A* is placed with its base at the origin and its tip at the point (2, 2). For each arrow listed below, identify whether it is congruent or not congruent to arrow *A*. For the congruent arrows, find the rigid motion or composition that shows the congruency.

a. base at (2, 2), tip at (0, 0)
b. base at (−2, 2), tip at (−4, 0)
c. base at (4, 0), tip at the origin
d. base at (1, 1), tip at (−1, −1)

MATHEMATICAL PRACTICES
Look for and Make Use of Structure

11. Isosceles right triangles *ABC* and $A'B'C'$ are positioned so that point *B* is the midpoint of $\overline{A'B'}$ and point B' is the midpoint of $\overline{BC}$.



a. Find a composition of rigid motions that shows the two triangles are congruent.
b. Is there more than one composition that shows congruence? If yes, identify another composition.
c. Could triangle ABA' be congruent to triangle *ABC*? Explain why or why not.