Concurrent Segments in Triangles

What's the Point?

Lesson 14-1 Altitudes of a Triangle

My Notes

Learning Targets:

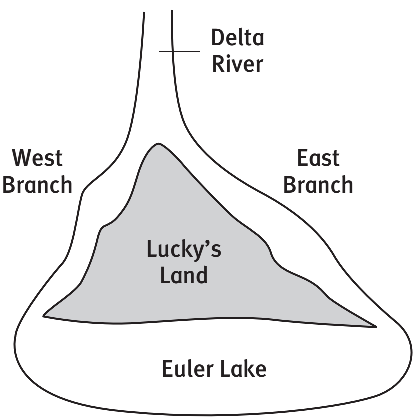
- Determine the point of concurrency of the altitudes of a triangle.
- Use the point of concurrency of the altitudes of a triangle to solve problems.

SUGGESTED LEARNING STRATEGIES: Summarizing, Paraphrasing, Interactive Word Wall, Vocabulary Organizer, Create Representations, Identify a Subtask, Use Manipulatives

Lucky Leon was an old gold miner whose hobby was mathematics. In his will, Leon left a parcel of land to be used as a nature preserve. The will included a map and the following clue to help find a buried treasure.

At the point on this map where two altitudes cross,
In a hole in the ground some treasure was tossed.
Stand at the place where two medians meet,
And the rest of the treasure will be under your feet.
Each locked in chests where they will stay
Until the time someone finds a way
To open each lock with a combination,
Using digits of the coordinates of each location.

Al Gebra, Geoff Metry, and Cal Lucas will design the layout of the nature preserve. They use the map to help with the design. The parcel is shaped like a triangle, with two rivers and a lake bordering the three sides.



In order to preserve the natural surroundings, Al, Geoff, and Cal want to build as few structures as possible. They decide that the design of the nature preserve will include:

- a visitor center at each of the three vertices of the triangle;
- a power station that is equidistant from the three visitor centers;
- a primitive campground inland, equidistant from each shoreline.

They also want to find Lucky Leon's buried treasure.

My Notes

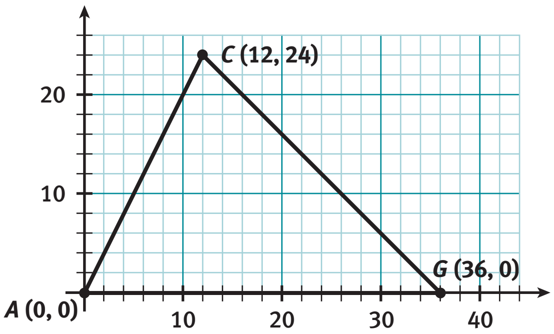
MATH TERMS

An **altitude of a triangle** is a segment from a vertex of the triangle, perpendicular to the opposite side (or line containing the opposite side) of the triangle.

Al, Geoff, and Cal decide that their first priority is to find the treasure. They begin with the first clue from Leon's poem.

At the point on this map where two altitudes cross,
In a hole in the ground some treasure was tossed.

Since they do not know which two ***altitudes*** Leon meant, Geoff decides to place a grid over the map and draw all three altitudes to find the coordinates of any points of intersection.



1. Determine the slopes of the three sides of $\triangle AGC$.

MATH TIP

Perpendicular lines have slopes that are opposite reciprocals. The slope of a horizontal line is *zero*. The slope of a vertical line is *undefined*.

2. Determine the slopes of the altitudes of the triangle.

a. What is the slope of the altitude from A to $\overline{CG}$? Justify your answer.

b. What is the slope of the altitude from C to $\overline{AG}$? Justify your answer.

c. What is the slope of the altitude from G to $\overline{AC}$? Justify your answer.

MATH TERMS

When three or more lines intersect at one point, the lines are said to be concurrent. The point where the three or more lines intersect is called the **point of concurrency**.

3. Use each vertex with its corresponding altitude's slope to graph the three altitudes of the triangle on the grid in Item 1.

4. After drawing the altitudes, Geoff is surprised to see that all three altitudes meet at one point. State the coordinates of the ***point of concurrency***.

My Notes

Al is not convinced that the altitudes of the triangle are concurrent and wants to use algebra to determine the coordinates of the points of intersection of the altitudes.

5. To use algebra to find the point where the altitudes meet, we need to know the equations of the altitudes.

a. State the coordinates of one point on the altitude from A to $\overline{CG}$.

b. Write the equation of the altitude from A to $\overline{CG}$.

c. Write the equation of the altitude from C to $\overline{AG}$.

d. Write the equation of the altitude from G to $\overline{AC}$.

6. Use the equations for the altitude from point A and the altitude from point C to determine the point of intersection. Show your work.

7. Verify that the point of intersection from Item 6 is also on the altitude from point G.

8. Explain why the algebra from Items 6 and 7 demonstrates that the three altitudes are concurrent.

Al reviews the first clue to the treasure in the poem:

At the point on this map where two altitudes cross,
In a hole in the ground some treasure was tossed.

9. Use appropriate tools strategically. Al is also not convinced that the altitudes are concurrent for every triangle. Use geometry software to draw a triangle and its three altitudes. Then drag the vertices to change the shape of the triangle.

a. Do the altitudes remain concurrent?

b. Is the point of concurrency always inside the triangle?

10. The first part of the buried treasure is located at the ***orthocenter*** of the triangle. What are the coordinates of the location of the first part of the treasure?

MATH TIP

Recall that the **point-slope form** for the equation of a line is $y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) is any point on the line.

The equation of a horizontal line is in the form $y = k$.

The equation of a vertical line is in the form $x = h$.

MATH TIP

There are several methods to solve a **system of equations** including the *substitution method* and the *elimination method* (also called linear combinations).

MATH TERMS

The point of concurrency of the altitudes of a triangle is called the **orthocenter**.

My Notes

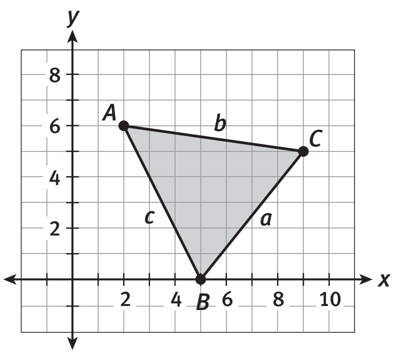
Check Your Understanding

11. Explain why two of the altitudes of a right triangle are its legs.

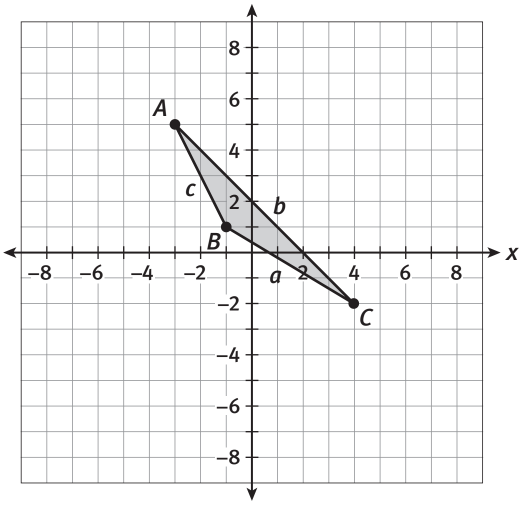
12. If one altitude of a triangle lies in the triangle's exterior and one lies in the triangle's interior, what is true about the location of the third altitude?

LESSON 14-1 PRACTICE

13. Graph the altitudes and the orthocenter of the triangle.



14. **Attend to precision.** Graph the altitudes and the orthocenter of the triangle.



Find the coordinates of the orthocenter of the triangle with the given vertices.

15. $A(-1, 2)$, $B(9, 2)$, $C(-1, -3)$

16. $X(0, 4)$, $Y(2, 2)$, $Z(0, -5)$

17. $D(-4, 9)$, $E(5, 6)$, $F(-2, 0)$

18. $A(8, 10)$, $B(5, -10)$, $C(0, 0)$

My Notes

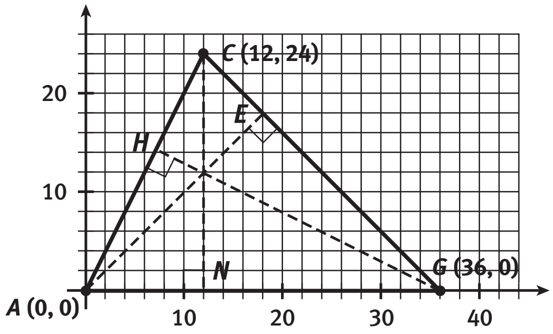
Learning Targets:

- Determine the point of concurrency of the medians of a triangle.
- Use the point of concurrency of the medians of a triangle to solve problems.

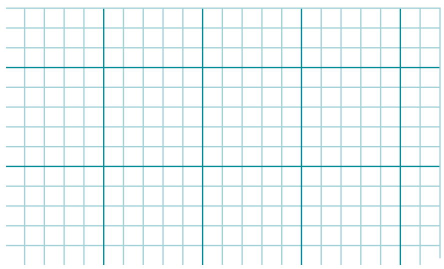
SUGGESTED LEARNING STRATEGIES: Summarizing, Paraphrasing, Interactive Word Wall, Vocabulary Organizer, Create Representations, Identify a Subtask, Use Manipulatives

The next two lines of the poem give the next clue:

Stand at the place where two medians meet,
And the rest of the treasure will be under your feet.



1. Carefully draw all three ***medians*** on $\triangle AGC$. Name the coordinates of the point(s) where the medians appear to cross. Do they appear to be concurrent?



MATH TERMS

A **median** of a triangle is a segment from a vertex of the triangle to the midpoint of the opposite side of the triangle.

Use concepts from algebra in Items 2–7 to prove or disprove your answer to Item 1.

2. Write the coordinates of the midpoints of each side of $\triangle AGC$. Label the midpoints as follows:

H is the midpoint of $\overline{AC}$.

N is the midpoint of $\overline{AG}$.

E is the midpoint of $\overline{CG}$.

3. Determine the equation of median $\overline{AE}$. Show your work.

My Notes

4. Determine the equation of median $\overline{GH}$. Show your work.

5. Determine the equation of median $\overline{CN}$. Show your work.

6. Determine the coordinates of the intersection of $\overline{AE}$ and $\overline{CN}$. Show your work.

7. Verify that the point of intersection from Item 6 is also on $\overline{GH}$.

8. Explain how the algebra demonstrates that the three medians are concurrent.

9. **Use appropriate tools strategically.** Al wants to be certain that the medians are concurrent for every triangle. Use geometry software to draw a triangle and its three medians. Then drag the vertices to change the shape of the triangle.
   a. Do the medians remain concurrent?

   b. Is the point of concurrency always inside the triangle?

The second part of the buried treasure is located at the ***centroid*** of the triangle.

MATH TERMS

The point of concurrency of the medians of a triangle is called the **centroid** of the triangle.

10. What are the coordinates of the location of the second part of the treasure? Label the centroid R in the diagram provided in Item 1.

The centroid's location has a special property related to segment lengths.

11. Use geometry software to draw the triangle and its medians shown in Item 1.

12. Use geometry software to find each length.

a. CR	**b.** RN	**c.** AR
d. RE	**e.** GR	**f.** RH

My Notes

13. Compare the lengths of the collinear segments. What relationship do you notice between the distance from the vertex to the centroid and the distance from the centroid to the midpoint of the opposite side?

14. The relationship you noticed in Item 13 can be stated as a theorem. Complete the **Centroid Measure Theorem:**

The centroid of a triangle divides each median into two parts so that the distance from the vertex to the centroid is ________________ the distance from the centroid to the midpoint of the opposite side.

The centroid is also the center of mass (balance point) of any triangle.

15. Create any triangle on construction paper. Locate the centroid. Cut out the triangle and balance the triangle on a pencil point placed under the centroid.

Check Your Understanding

16. Can an altitude of a triangle also be a median of the triangle? Explain.

Tell whether each statement is *always*, *sometimes*, or *never* true. Draw a sketch to support your answer.

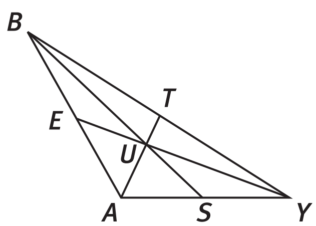
17. The medians of a triangle bisect each of the angles of the triangle.

18. The centroid of a triangle lies in the interior of the triangle.

19. The centroid of an equilateral triangle is equidistant from each of the vertices.

LESSON 14-2 PRACTICE

Given: $\triangle BAY$ with centroid U. Complete the following.



20. If $EU = 8$ cm, then

$UY =$ ________ and $YE =$ ________.

21. If $BS = 12$ cm, then

$BU =$ ________ and $US =$ ________.

22. **Reason abstractly.** If $TU = 10x$ cm, then

$AU =$ ________ and $AT =$ ________.

Find the coordinates of the centroid of the triangle with the given vertices.

23. $A(0, -4)$, $B(1, 1)$, $C(-2, 6)$

24. $X(6, 0)$, $Y(2, 8)$, $Z(-2, -2)$

25. $D(-4, -2)$, $E(1, 0)$, $F(9, 5)$

26. $A(8, 6)$, $B(3, -1)$, $C(0, 7)$

My Notes

Learning Targets:

- Determine the points of concurrency of the perpendicular bisectors and the angle bisectors of a triangle.
- Use the points of concurrency of the perpendicular bisectors and the angle bisectors of a triangle to solve problems.

SUGGESTED LEARNING STRATEGIES: Summarizing, Paraphrasing, Interactive Word Wall, Vocabulary Organizer, Create Representations, Identify a Subtask, Use Manipulatives

The third part of the poem tells how to determine the combination of the lock:

Each locked in chests where they will stay
Until the time someone finds a way
To open each lock with a combination,
Using digits of the coordinates of each location.

1. . . . *the coordinates of each location* refers to the coordinates of the orthocenter and the centroid you found in the previous lessons. If the combination consists of three numbers that are digits of the coordinates of the location of the treasure, list some possible combinations for the lock on the second treasure chest.

MATH TERMS

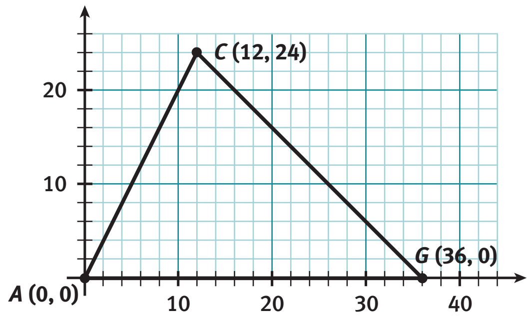
The perpendicular bisectors of the sides of a triangle are concurrent. This point of concurrency is called the **circumcenter**.

MATH TERMS

The angle bisectors of the angles of a triangle are concurrent, and this point of concurrency is called the **incenter** of the triangle.

Al, Geoff, and Cal found the treasure by locating the point of concurrency of the altitudes of a triangle, called the orthocenter, and the point of concurrency of the medians of a triangle, called the centroid. Cal tells Al and Geoff to learn about two more types of concurrent points, called the ***circumcenter*** and the ***incenter***, before they design the nature preserve. They start by locating the coordinates of the circumcenter of $\triangle AGC$.

2. Al took an algebraic approach similar to the manner in which he found the orthocenter and centroid of the triangle. He calculated the circumcenter to be (18, 6). Geoff decided to carefully sketch the *perpendicular bisector* of each side of the triangle. Carefully draw the perpendicular bisector of each side of the triangle and label the point of concurrency Q.



Does your drawing verify that Al is correct?

The circumcenter of a triangle has a special property.

3. Use the distance formula to complete the following table.

Distance from Q to A	
Distance from Q to C	
Distance from Q to G	

My Notes

4. The distances suggest that the circumcenter of a triangle is equidistant from the three vertices of the triangle. Write a convincing argument that this special property applies to all triangles.

5. Sketch the ***circumscribed circle*** on the grid in Item 2.

MATH TERMS

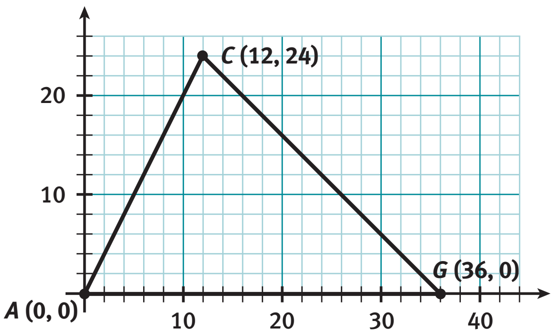
A **circumscribed circle** is a circle that contains all the vertices of a polygon.

Cal located the incenter of $\triangle AGC$. He estimated that the coordinates of the incenter are approximately (14.446, 8.928).

6. On the grid below, carefully draw the three bisectors of the angles of $\triangle AGC$. Label the point of concurrency I. Does your drawing support Cal's results?

MATH TIP

Since the circumcenter is equidistant from the three vertices of the triangle, it is the center of the circle with radius equal to the distance from the circumcenter to any of the vertices.



To explore a special property of the incenter, Cal needs to calculate the distances from the incenter to each of the three sides.

7. a. Draw a perpendicular segment from I to $\overline{AC}$ on the triangle in Item 6. Label the point of intersection L.
b. Cal estimated the coordinates of L to be (6.177, 12.353). Find the length IL.

8. a. Draw a perpendicular segment from I to $\overline{AG}$ on the triangle in Item 6. Label the point of intersection M.
b. Estimate the coordinates of M.

c. Find the length IM.

MATH TIP

The distance from a point to a line (or segment) is the length of the perpendicular segment from the point to the line.

9. a. Draw a perpendicular segment from I to $\overline{CG}$ on the triangle in Item 6. Label the point of intersection B.
b. Estimate the coordinates of B.

c. Find the length IB.

MATH TERMS

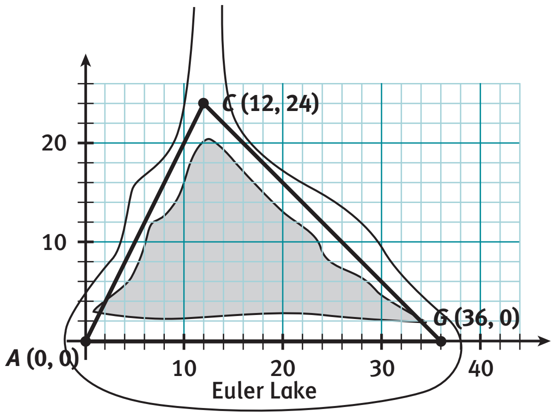
Since the **incenter** is equidistant to the three sides of the triangle, it is the center of the circle with radius equal to the distance from the incenter to any of the sides. This is called the **inscribed circle**.

The distances suggest that the incenter of a triangle is equidistant to the three sides of the triangle.

10. Sketch the ***inscribed circle*** on the triangle in Item 6.

My Notes

11. Reason quantitatively. Refer back to the description of the nature preserve at the beginning of this Activity. On the map below, locate and label the three visitor centers, the power station, and the campground. Explain why you chose each location.



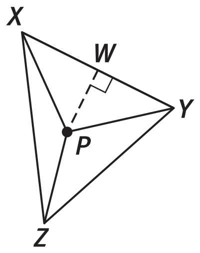
Check Your Understanding

12. Can the incenter of a triangle lie outside the triangle? Explain.

13. Draw three sketches showing the circumcenter lying inside, outside, and on the triangle. Make a conjecture about the location of a circumcenter and the type of triangle.

LESSON 14-3 PRACTICE

Given: $\triangle XYZ$ with circumcenter P.



Complete the following.

14. If $PX = 18$ mm, then

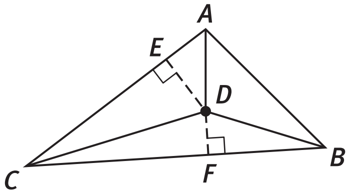
$PY =$ ________ and $PZ =$ ________.

15. If $WX = 12$ cm, then $XY =$ ________.

Given: $\triangle ABC$ with incenter D. Complete the following.

16. If $DE = 3$ cm, then $DF =$ ________.

17. Make sense of problems. If and $m\angle CAB = 90°$ and $m\angle CBA = 25°$, $m\angle ACB =$ ________ and $m\angle ACD =$ ________.



Find the coordinates of the circumcenter of the triangle with the given vertices.

18. $A(-3, 0)$, $B(0, 0)$, $C(0, 20)$

19. $X(2, 1)$, $Y(-5, -3)$, $Z(0, 7)$

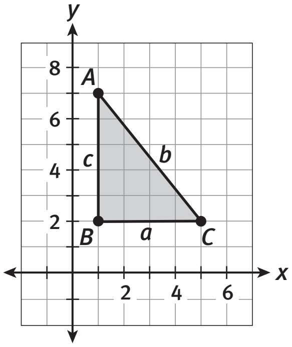
ACTIVITY 14 PRACTICE

Write your answers on notebook paper. Show your work.

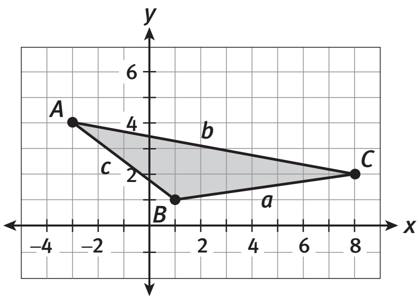
Lesson 14-1

Graph the altitudes and find the orthocenter of each triangle.

1.



2.



Find the coordinates of the orthocenter of the triangle with the given vertices.

3. $A(-1, 2)$, $B(9, 2)$, $C(-1, -3)$

4. $X(0, 8)$, $Y(3, 5)$, $Z(2, -2)$

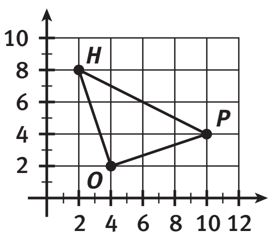
5. $D(-4, 9)$, $E(5, 6)$, $F(-2, 0)$

6. $A(1, 1)$, $B(4, -1)$, $C(0, 2)$

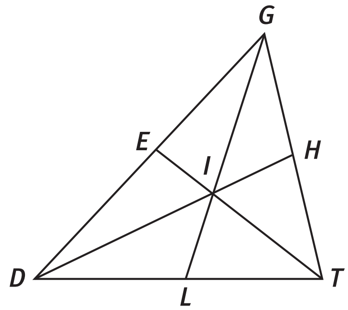
7. Are the altitudes of an equilateral triangle also its lines of symmetry? Explain.

Lesson 14-2

8. Draw the centroid of $\triangle HOP$ and name its coordinates.



For Items 9 and 10, use $\triangle DTG$ with centroid I.

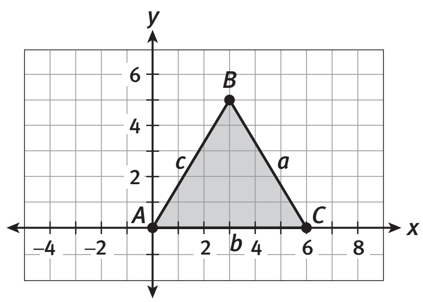


9. If $GI = 24$ cm, then $LI =$ ________ and $LG =$ ________.

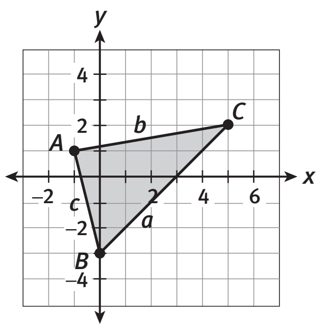
10. If $DH = (4x + 10)$ in. and $HI = (2x - 4)$ in., then $x =$ ________, $HI =$ ________, and $ID =$ ________.

Graph the medians and find the centroid of each triangle.

11.



12.



Find the coordinates of the centroid of the triangle with the given vertices.

13. $A(8, 10)$, $B(4, -10)$, $C(0, 0)$

14. $X(0, 5)$, $Y(3, 12)$, $Z(1, -5)$

15. $D(-4, 9)$, $E(5, 6)$, $F(-1, 0)$

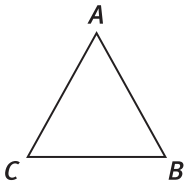
16. $A(8, 10)$, $B(1, -10)$, $C(3, 3)$

17. Use geometry software to draw an equilateral triangle. Draw the altitudes and the medians of the triangle. Make a conjecture about the orthocenter and centroid of an equilateral triangle. Write a two-column proof to prove your conjecture or find a counterexample to show it is false.

If ______________________________________,
then ______________________________.

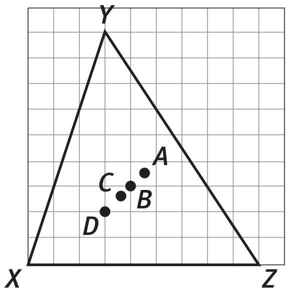
Given: Equilateral $\triangle ABC$ with ________________

Prove: ________________



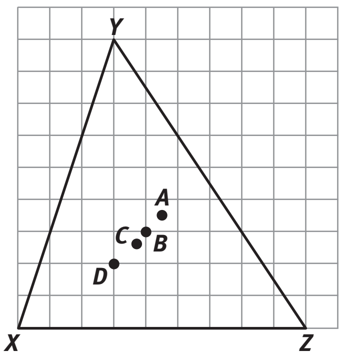
Lesson 14-3

Points A, B, C, and D represent different points of concurrency of segments associated with $\triangle XYZ$. The coordinates are $X(0, 0)$, $Z(9, 0)$, and $Y(3, 9)$.



18. Identify A, B, C, and D as the orthocenter, centroid, circumcenter, or incenter of the triangle. Explain your reasoning.

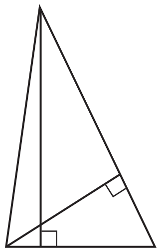
19. Suppose that points X, Y, and Z are the locations of three rustic cabins with no running water. If the cabins will be supplied with water from a single well, where would you locate the well? Explain your reasoning.



20. Three sidewalks cross near the middle of City Park, forming a triangular region. Levi donated money to the city to create a fountain in memory of his father. He wants the fountain to be located at a place that is equidistant from the three sidewalks. At which point of concurrency should it be located?

A. orthocenter **B.** centroid
C. circumcenter **D.** incenter

21. Which of the points of concurrency is illustrated?



A. orthocenter
B. centroid
C. circumcenter
D. incenter

Find the coordinates of the circumcenter of the triangle with the given vertices.

22. $A(-3, -1)$, $B(-4, -5)$, $C(0, -1)$

23. $X(7, 4)$, $Y(-2, -7)$, $Z(1, 2)$

MATHEMATICAL PRACTICES
Reason Abstractly and Quantitatively

24. a. Which point(s) of concurrency could lie outside of a triangle?

b. What type(s) of triangle would cause that to happen?

c. Sketch the incenter of $\triangle NET$.